Università degli Studi di Pisa
Dottorato di Ricerca in Matematica - XII Ciclo

Integral Duality and the structure of Tate Cohomology Rings

Candidato: Antonio Bellezza
Relatore: René Schoof
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Overview

Section I is an introduction to the subject of group cohomology. We recall some of the main results in the theory of ordinary and Tate cohomology modules and rings.

In **Section II**, we get some results on the structure of Tate cohomology rings and modules:

- We prove that unlike in the ordinary cohomology case, $\widehat{H}^*(G, \mathbf{Z})$ is not noetherian unless G has periodic cohomology. Furthermore, all elements of negative degree are nilpotent.
- We prove that the multiplicative structures of $\widehat{H}^*(G, \mathbf{Z})$ and $\widehat{H}^*(G, \mathbf{Z}/n\mathbf{Z})$ are uniquely identified by the additive structure of the Tate cohomology module, the cup product for the ordinary cohomology and a finite number of products in negative degree. This allows the extension to Tate cohomology of results known for the ordinary cohomology case, if one is able to compute products in negative degrees.
- We compute the ring structure of $\widehat{H}^*(\mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}/p^b\mathbf{Z}, \mathbf{Z})$ for p an odd prime.
- We compute the structure of $\widehat{H}^*(\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ both as a ring and as an algebra over $\widehat{H}^*(\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}, \mathbf{Z})$.
- For any abelian p-group G, with p odd, we exhibit the cup products in $\widehat{H}^*(G, \mathbf{Z})$ for all but a finite number of couples of degrees.

Section III deals with the problems arising when one wants to implement a program computing cohomology and cup products. We show some possible solutions and describe an actual implementation. We describe the results of some experiments.

Appendix A contains the output from some runs of the software. Complete resolutions, cohomology modules and cup products are computed for some small abelian groups and degrees. Timings are also given.

I : Introduction to Tate cohomology

I.1 : G-modules

To help fix some notation, we recall here the definitions and some properties of the basic objects we are going to be working with. This is not meant to be an exhaustive introduction. For that, see [C–E], [Br], [Ser] or [Wei].

If G is a group, a *left G-module M* (or simply G-module) is an abelian group with a left G-action, i.e. a morphism from G to the group of automorphisms of M. That means that if g, h are elements of the group G, e is the neutral element of the group and a, b are elements of M, one has

$$\begin{aligned} g(a+b) &= g(a) + g(b) \\ gh(a) &= g(h(a)) \\ e(a) &= a. \end{aligned}$$

A *G-submodule N* (or simply *submodule* if no confusion can arise) of a G-module M is a subgroup of M which is invariant under the action of G, i.e. such that $g(N) \subset N$ for all $g \in G$. Examples are the zero submoodule $\{0\}$ and the submodule of invariants

$$M^G := \{m \in M \mid g(m) = m \text{ for any } g \in G\}.$$

The quotient of a G-module by a G-submodule is itself a G-module with the induced action.

An abelian group (also called *module*) is a G-module for $G = \{e\}$. A morphism of G-modules is a homomorphism of the underlying abelian groups which commutes with the G-action. If ϕ is the morphism, g a group element and M and N are the domain and codomain respectively, then the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow g & & \downarrow g \\ M & \xrightarrow{\phi} & N \end{array}$$

commutes, i.e. $g(\phi(m)) = \phi(g(m))$ for any $g \in G$ and $m \in M$.

The set of morphisms of G-modules from M to N, which we will also call indifferently maps of G-modules or G-maps, is denoted by $\mathrm{Hom}_G(M, N)$, while we denote by $\mathrm{Hom}(M, N)$ the set of morphisms of abelian groups from M to N. Observe that $\mathrm{Hom}_G(M, N) \subset \mathrm{Hom}(M, N)$.

$\mathrm{Hom}(M, N)$ becomes a G-module in the following way: if $f \in \mathrm{Hom}(M, N)$, then

$$g(f) = g \circ f \circ g^{-1}.$$

$\mathrm{Hom}_G(M, N)$ is a submodule of $\mathrm{Hom}(M, N)$. It actually equals $\mathrm{Hom}(M, N)^G$. In particular, the induced action of G is trivial.

For a group G, the set of G-modules with G-morphisms forms an abelian category.

We now introduce an object which will play an important role in what follows: the *group ring of* G, denoted $\mathbf{Z}[G]$, is the ring which is generated as a free $\mathbf{Z}$-module by elements (g) for g varying in G. The product on additive generators is inherited by the group operation in the following way:

$$(g) \cdot (h) = (gh).$$

The element (e) is the identity element of the ring. The ring is commutative if and only if G is. If M is a G-module, the G-action can be extended by $\mathbf{Z}$-linearity to a ring action of $\mathbf{Z}[G]$ on M. On the other hand, given a ring action of $\mathbf{Z}[G]$ on an abelian group M, the set inclusion

$$G \hookrightarrow \mathbf{Z}[G]$$
$$g \longmapsto (g)$$

induces a group action of G on M. Thus, being a G-module is equivalent to being a $\mathbf{Z}[G]$-module.

If G is finite, then the element $N = \sum_{g \in G} (g)$ of $\mathbf{Z}[G]$ is called the *norm element*. It acts on an element m of a G-module M as the usual G-norm.

I.2 : Group cohomology

If M is a G-module, then $\mathrm{Hom}_G(\_, M)$ is a left-exact contravariant functor from the category of G-modules to itself. That means that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of G-modules, then

$$0 \longrightarrow \mathrm{Hom}_G(C, M) \longrightarrow \mathrm{Hom}_G(B, M) \longrightarrow \mathrm{Hom}_G(A, M)$$

is exact.

A *projective G-module* is a direct summand of a free G-module. Projective modules are characterized by the following property: given a diagram of G-modules and G-maps

$$\begin{tikzcd} & P \arrow[d] \arrow[dr, "0"] & \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

with P projective and exact bottom row, then there exists a G-map f such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[d] \arrow[dr, "0"] & \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

A *projective resolution* $\varepsilon : P_* \longrightarrow \mathbf{Z}$ is an exact sequence of G-modules of the form

$$\ldots \longrightarrow P_i \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{Z} \longrightarrow 0$$

with P_i's projective and the trivial action of G on $\mathbf{Z}$. Given such a resolution and a G-module M, the sequence

$$0 \longrightarrow \mathrm{Hom}_G(P_0, M) \longrightarrow \mathrm{Hom}_G(P_1, M) \longrightarrow \mathrm{Hom}_G(P_2, M) \longrightarrow \ldots$$

is not necessarily exact, but is a cochain complex, i.e. the composition of two consecutive maps is zero. The cohomology at place i is called the i-th *cohomology module of G with coefficients in M* and is denoted by $H^i(G, M)$. We denote $H^*(G, M)$ the graded module $\bigoplus_{i \geq 0} H^i(G, M)$. Cohomology modules don't depend on the choice of the resolution and free resolutions (which are in particular projective) always exist. For instance, such a resolution can be constructed step by step. It is possible to start with $P_0 = \mathbf{Z}[G]$ and ε the augmentation map sending an element $\sum_i c_i(g_i)$ of $\mathbf{Z}[G]$ to the sum $\sum_i c_i$ of the coefficients. Once one has the resolution up to P_i, for P_{i+1} it is possible to take any free $Z[G]$-module mapping onto the kernel of $P_i \longrightarrow P_{i-1}$.

An explicit resolution is the *standard resolution*

$$\dots \xrightarrow{d_3} \mathrm{Sr}_2 \xrightarrow{d_2} \mathrm{Sr}_1 \xrightarrow{d_1} \mathrm{Sr}_0 \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0.$$

It is constructed as follows:

- The i-th object Sr_i is the group ring $\mathbf{Z}[G^{i+1}]$ of the $i+1$-fold direct product of G.
- The map ε is the augmentation map, as described above.
- The boundary map d_i sends element $(g_0, \dots, g_i)$ to $\sum_{j=0}^{i}(-1)^j(g_0, \dots, \widehat{g}_j, \dots, g_i)$, where the hat notation denotes as usual the element to be skipped.

The module Sr_i is $\mathbf{Z}[G]$-free, as

$$\mathrm{Sr}_i = \bigoplus_{g_1,\dots,g_i \in G} \mathbf{Z}[G](e, g_1, \dots, g_i).$$

As the sequence

$$0 \longrightarrow \mathrm{Hom}_G(\mathbf{Z}, M) \longrightarrow \mathrm{Hom}_G(P_0, M) \longrightarrow \mathrm{Hom}_G(P_1, M)$$

is exact, the module $H^0(G, M)$ is isomorphic to $\mathrm{Hom}_G(\mathbf{Z}, M)$, which is in turn isomorphic to M^G via the map sending $f \in \mathrm{Hom}_G(\mathbf{Z}, M)$ to $f(1)$.

For each $i \geq 0$, $H^i(G, \_)$ is a covariant functor from the category of G-modules to the category of $\mathbf{Z}$-modules. In fact, a G-map $f : A \longrightarrow B$ induces maps of cochain complexes

$$\begin{aligned} \mathrm{Hom}_G(P_*, A) &\xrightarrow{f^*} \mathrm{Hom}_G(P_*, B) \\ a &\longmapsto f \circ a \end{aligned}$$

and given two G-maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$, one has the equality

$$(g \circ f)^* = g^* \circ f^*.$$

Being maps of cochain complexes, they induce maps on the cohomology modules satisfying the same composition property.

Given a short exact sequence of G-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

one has the corresponding short exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Hom}_G(P_*, A) \xrightarrow{f^*} \mathrm{Hom}_G(P_*, B) \xrightarrow{g^*} \mathrm{Hom}_G(P_*, C) \longrightarrow 0$$

so the functors $H^i(G, \_)$ fit the long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(G,A) \xrightarrow{f^*} H^0(G,B) \xrightarrow{g^*} H^0(G,C) \xrightarrow{\delta_0} \dots \\ \dots \xrightarrow{\delta_0} H^1(G,A) \xrightarrow{f^*} H^1(G,B) \xrightarrow{g^*} H^1(G,C) \xrightarrow{\delta_1} \dots \\ \dots \xrightarrow{\delta_1} H^2(G,A) \xrightarrow{f^*} H^2(G,B) \xrightarrow{g^*} H^2(G,C) \xrightarrow{\delta_2} \dots \end{array}$$

This makes the family of functors $H^i(G, \_)$ a *cohomological functor.*

If $H < G$ is a subgroup, then any G-module is also an H-module. A projective G-module is also projective as an H-module. Thus, any G-projective resolution is also an H-projective resolution. Calling $\mathrm{Coind}_H^G M$ the coinduced module $\mathrm{Hom}_H(\mathbf{Z}[G], M)$, with the G-action inherited by the action on $\mathrm{Hom}(\mathbf{Z}[G], M)$, if $\varepsilon : P_* \longrightarrow \mathbf{Z}$ is a complete G-projective resolution one has the isomorphism of cochain-complexes

$$\mathrm{Hom}_H(P_*, M) \cong \mathrm{Hom}_G(P_*, \mathrm{Coind}_H^G M).$$

Passing to cohomology, this gives the following:

Theorem I.2.1(Shapiro's lemma): *Let $H < G$ be groups and let M be a H-module. Then*

$$H^*(H, M) \cong H^*(G, \mathrm{Coind}_H^G M).$$

For finite fields, one can develop a cohomology theory with integral (including negative) degrees. A *complete resolution* $\varepsilon : P \longrightarrow \mathbf{Z}$ is an exact sequence

$$\dots \longrightarrow P_i \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{d_o} P_{-1} \longrightarrow P_{-2} \longrightarrow \dots$$

where the P_i's are projective, together with a surjection $\varepsilon : P_0 \longrightarrow \mathbf{Z}$ such that d_0 factors as

$$P_0 \xrightarrow{\varepsilon} \mathbf{Z} \hookrightarrow P_{-1}.$$

Let $\varepsilon : P_* \longrightarrow \mathbf{Z}$ be a projective resolution with finitely generated P_i's. Consider the *dual resolution*

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\hat{\varepsilon}} \mathrm{Hom}(P_0, \mathbf{Z}) \longrightarrow \mathrm{Hom}(P_1, \mathbf{Z}) \longrightarrow \mathrm{Hom}(P_2, \mathbf{Z}) \longrightarrow \dots$$

It is possible to get a complete resolution joining them in the following way:

$$\begin{array}{ccccc} \dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow & P_0 & \xrightarrow{d_0} & \mathrm{Hom}(P_0,\mathbf{Z}) & \longrightarrow \mathrm{Hom}(P_1,\mathbf{Z}) \longrightarrow \\ & \downarrow\varepsilon & & \uparrow\hat{\varepsilon} & \\ & \mathbf{Z} & = & \mathbf{Z} & \end{array}$$

where d_0 is defined so that the square commutes.

If $P_0 \cong \mathbf{Z}[G]$, then $\mathrm{Hom}(P_0,\mathbf{Z}) \cong \mathbf{Z}[G]$ and d_0 is the norm map, i.e. multiplication by element $N = \sum_{g\in G}(g) \in \mathbf{Z}[G]$.

The i-th cohomology of the cochain complex $\mathrm{Hom}_G(P_*, M)$ is called the *i-th Tate cohomology module of G with coefficients in M*. The $\mathbf{Z}$-graded module

$$\widehat{H}^*(G,M) = \bigoplus_{i\in\mathbf{Z}} \widehat{H}^i(G,M),$$

is called the *Tate cohomology module*.

One has the following isomorphisms:

$$\begin{aligned} \widehat{H}^0(G,M) &\cong \frac{M^G}{N(M)} \\ \widehat{H}^{-1}(G,M) &\cong \frac{\mathrm{Ker} N}{\sum_{g\in G}((g)-(e))M}. \\ \widehat{H}^i(G,M) &\cong H^i(G,M) \quad \text{if } i \geq 1. \end{aligned}$$

I.3 : Cup product

If G is a finite group, given two G-modules M and N, there are cup-product operations

$$\begin{array}{ccccll} H^i(G,M) & \otimes & H^j(G,N) & \xrightarrow{\cup} & H^{i+j}(G,M\otimes N) & \text{for } i,j \geq 0 \\ \widehat{H}^i(G,M) & \otimes & \widehat{H}^j(G,N) & \xrightarrow{\cup} & \widehat{H}^{i+j}(G,M\otimes N) & \text{for } i,j \in \mathbf{Z} \end{array}$$

where the G-action on $M \otimes N$ is the diagonal one

$$g(m\otimes n) = gm \otimes gn.$$

For non-negative degrees, the cup product of Tate cohomology is induced by the product for ordinary cohomology via the maps

$$H^i(G,M) \xrightarrow{\cong} \widehat{H}^i(G,M)$$

for $i > 0$ and

$$M^G \longrightarrow \frac{M^G}{NM}$$

at degree 0.

The cup product is identified by the following properties, which we express only for ordinary cohomology but which carry through to Tate cohomology:

Property I.3.1(degree 0): *The cup product*

$$H^0(G,M) \otimes H^0(G,N) \longrightarrow H^0(G, M \otimes N)$$

is the map

$$M^G \otimes N^G \longrightarrow (M \otimes N)^G$$

induced by the inclusions $M^G \hookrightarrow M$ and $N^G \hookrightarrow N$.

Property I.3.2(compatibility with δ): *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of G-modules and let M be a G-module such that the sequence

$$0 \longrightarrow A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is exact. Then

$$\delta(u \cup v) = \delta u \cup v$$

for any $u \in H^i(G,C)$ and any $v \in H^j(G,M)$.

Property I.3.3(graded-commutativity): *Let M and N be G-modules. Then for any $u \in H^i(G,M)$, $v \in H^j(G,N)$ we have*

$$u \cup v = (-1)^{ij} \phi^*(v \cup u),$$

where ϕ^* *is the map induced on cohomology by the canonical isomorphism*

$$M \otimes N \xrightarrow{\phi} N \otimes M.$$

Other important properties are the following:

Property I.3.4(naturality): *Let* $f : M \longrightarrow M'$ *and* $g : N \longrightarrow N'$ *be maps of* G*-modules. For any* $u \in H^i(G, M)$, $v \in H^j(G, N)$, *we have*

$$f^*(u) \cup g^*(v) = (f \otimes g)^*(u \cup v).$$

Property I.3.5(associativity): *Let* A, B *and* C *be* G*-modules and let* $a \in H^*(G, A)$, $b \in H^*(G, B)$ *and* $c \in H^*(G, C)$. *Then*

$$(a \cup b) \cup c = a \cup (b \cup c) \in H^*(G, A \otimes B \otimes C).$$

Tate cohomology possesses one property linking the modules with positive and negative degrees.

Property I.3.6(integral duality): *Let* M *be a* G*-module and call*

$$M' = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z}).$$

Consider the map

$$\begin{aligned} M \otimes M' &\longrightarrow \mathbf{Q}/\mathbf{Z} \\ m \otimes f &\longmapsto f(m) \end{aligned}$$

The induced map, composed with the cup product, gives rise to maps

$$\widehat{H}^i(G, M) \otimes \widehat{H}^{-i-1}(G, M') \longrightarrow \widehat{H}^{-1}(G, \mathbf{Q}/\mathbf{Z}) \cong \widehat{H}^0(G, \mathbf{Z}) \cong \mathbf{Z}/\#G\mathbf{Z}.$$

These maps are duality pairings, i.e. they induce isomorphisms

$$\widehat{H}^i(G, M) \cong \mathrm{Hom}(\widehat{H}^{-i-1}(M'), \widehat{H}^{-1}(G, \mathbf{Q}/\mathbf{Z}))$$

and

$$\widehat{H}^{-i-1}(G, M') \cong \mathrm{Hom}(\widehat{H}^i(G, M), \widehat{H}^{-1}(G, \mathbf{Q}/\mathbf{Z}))$$

If $M = \mathbf{Z}$, then integral duality has also the following form:

Property I.3.7(integral duality for Z): *Let G be a finite group acting trivially on $\mathbf{Z}$. Then the cup product*

$$\widehat{H}^i(G,\mathbf{Z}) \otimes \widehat{H}^{-i}(G,\mathbf{Z}) \longrightarrow \widehat{H}^0(G,\mathbf{Z})$$

is a duality pairing for any $i \in \mathbf{Z}$.

I.4 : Cohomology rings

When the module of coefficients is a ring on which G acts as a group of automorphisms, the cup product provides Tate and ordinary cohomology modules with a structure of graded-commutative rings. We will only develop the theory for the commutative case with trivial G-action, but the reader should keep in mind that generalizations are possible.

Let R be a commutative ring (with unity) on which G acts trivially. The product map

$$\begin{aligned} R \otimes R &\longrightarrow R \\ a \otimes b &\longmapsto ab \end{aligned}$$

is a map of G-modules, so it induces a map

$$\widehat{H}^*(G, R \otimes R) \longrightarrow \widehat{H}^*(G,R).$$

The composed map

$$\widehat{H}^*(G,R) \otimes \widehat{H}^*(G,R) \stackrel{\cup}{\longrightarrow} \widehat{H}^*(G, R \otimes R) \longrightarrow \widehat{H}^*(G,R),$$

which we still call cup product and denote with the $\cup$ symbol, provides $\widehat{H}^*(G,R)$ with a structure of graded-commutative ring, i.e. for any two homogeneous elements $\alpha \in \widehat{H}^i(G,R)$ and $\beta \in \widehat{H}^j(G,R)$ one has $\alpha\beta = (-1)^{ij}\beta\alpha$. This is the *Tate cohomology ring with ring of coefficients R*. The cohomology class of the identity element of R becomes the identity element in the ring $\widehat{H}^*(G,R)$ and the class of an invertible element of R becomes invertible in $\widehat{H}^*(G,R)$, since the cup product on $\widehat{H}^0(G,R)\otimes\widehat{H}^0(G,R)$ is induced by the product in R.

If M is a G-module which is also an R-module and if $g(rm)=rg(m)$ for all $g\in G$, $r\in R$ and $m\in M$, then the action map

$$\begin{aligned} R\otimes M &\longrightarrow M \\ r\otimes m &\longmapsto rm \end{aligned}$$

is a G-map. Passing to cohomology, this makes $\widehat{H}^*(G,M)$ a graded $\widehat{H}^*(G,R)$-module.

If R is $\mathbf{Z}$ with trivial G-action, the ring $\widehat{H}^*(G,\mathbf{Z})$ acts on the Tate cohomology of any G-module M. Thus $\widehat{H}^*(G,\mathbf{Z})$ appears as a rather natural object to study. Its structure provides us with valuable information. As an example of this assertion, the fact that $\widehat{H}^0(G,\mathbf{Z})$ is cyclic with order $\#G$ implies that $\#G$ annihilates $\widehat{H}^*(G,M)$ for any G-module M. As another example, if $\widehat{H}^*(G,\mathbf{Z})$ contains a homogeneous invertible element α of degree d, then, for any G-module M and any $i\in\mathbf{Z}$, cupping with α gives an isomorphism

$$\widehat{H}^i(G,M)\xrightarrow{\cup\alpha}\widehat{H}^{i+d}(G,M).$$

If $d\neq 0$, we say that such a group has *periodic cohomology* and we have the following result (see [Br], theorem 9.5):

Lemma I.4.1: *The following are equivalent for a finite group G:*

1. *G has periodic cohomology;*
2. *there exist integers n and d, with $d\neq 0$, such that $\widehat{H}^n(G,M)\cong\widehat{H}^{n+d}(G,M)$ for all G-modules M;*
3. *$\widehat{H}^d(G,\mathbf{Z})\cong\mathbf{Z}/\#G\mathbf{Z}$ for some $d\neq 0$;*
4. *$\widehat{H}^d(G,\mathbf{Z})$ contains an element of order $\#G$ for some $d\neq 0$;*
5. *every abelian subgroup of G is cyclic.*

Any finite cyclic group has periodic cohomology. In fact, if g is a generator of $G\cong\mathbf{Z}/n\mathbf{Z}$ and N is the norm element in $\mathbf{Z}[G]$, then

$$\ldots\longrightarrow\mathbf{Z}[G]\xrightarrow{(g)-(e)}\mathbf{Z}[G]\xrightarrow{N}\mathbf{Z}[G]\xrightarrow{(g)-(e)}\mathbf{Z}[G]\xrightarrow{N}\mathbf{Z}[G]\longrightarrow\ldots$$

is a complete resolution and one has

$$\widehat{H}^i(G,M)\cong\widehat{H}^{i+2}(G,M)$$

for any G-module M and any integer i.

We denote by G_p a p-Sylow subgroup of a finite group G and the p-part of a module A by $A_p = \bigcup_{i\geq 1} \mathrm{Ker}(p^i)$. If A is a ring, its p-parts are ideals. In case A has finite exponent, it is isomorphic (both as a module and as a ring) to the product of its p-parts. This is the case, for instance, for Tate cohomology modules and rings, for any module of coefficients, as the cardinality of the group annihilates all elements in these cases. Some cohomological properties can be studied by addressing the problems for p-groups. For instance, it follows at once from Lemma I.4.1 5 that a group G has periodic cohomology if and only if its Sylow subgroups do. We say that a group has *p-periodic cohomology* if its p-Sylow subgroup has periodic cohomology.

The previous discussion, *mutatis mutandis*, is also valid for ordinary cohomology.

II : Integral duality and Tate cohomology rings

II.1 : Introduction

Group cohomology is a tool used in several settings. Some objects arising from the action of a group on a module can be interpreted as cohomology modules. This makes a wide set of tools available. For instance, if G is the Galois group of a finite Galois extension of fields $K \subset L$, then G acts on objects such as

- The additive group $(L, +)$;
- The multiplicative group $(L^*, \times)$;
- The group of units $\mathcal{O}_L$, in case K and L are algebraic number fields or local fields.

If M is one of these G-modules, then $H^0(G, M)$ is $M \cap K$ and $N \in \mathbf{Z}[G]$ acts as the usual norm if M is a subgroup of $(L^*, \times)$ or as the trace map if M is a subgroup of $(L, +)$. For instance, $\widehat{H}^0(G, L^*) \cong K^*/N(L^*)$ is the group of elements of K^* modulo the norms of elements in L^*.

If the group G is cyclic, then the cohomology is periodic and to compute the module structure of $\widehat{H}^*(G, M)$ it is enough to compute the cohomology modules for two consecutive degrees. For instance, if $\widehat{H}^i(G, M) \cong \mathbf{Z}/\#G\mathbf{Z}$ and $\widehat{H}^{i+1}(G, M) = \{0\}$, then $\widehat{H}^*(G, M)$ is isomorphic to $\widehat{H}^*(G, \mathbf{Z})$, the isomorphism being given by cup product with an element of $\widehat{H}^i(G, M)$ of order $\#G$. Thus, the structure of cohomology rings can provide valuable information on cohomology modules.

Ordinary cohomology rings are widely studied and many results on their multiplicative structure are available. Among the most relevant results, we quote

Proposition II.1.1([Ev]): *If G is a finite group and R is finitely generated, then the ordinary cohomology ring $H^*(G,R)$ is a finitely generated ring.*
If M is a noetherian module over R, then $H^*(G,M)$ is noetherian over $H^*(G,R)$.

Proposition II.1.2([Chap]): *If G is a finite abelian p-group, with p an odd prime, then the rings $H^*(G,\mathbf{Z})$ and $H^*(G,\mathbf{Z}/p^i\mathbf{Z})$ (for p^i at least the exponent of the group) can be expressed in explicit form in terms of generators and relations (given in the paper).*

Computations with the use of a computer have been carried out to yield the structure of $\widehat{H}^*(G,\mathbf{F}_2)$ for 2-groups ([Rus], [Car–3], [Car–1], [Whi]).

For the Tate case fewer explicit results are available. An important property for the study of Tate cohomology is integral duality: the maps

$$\widehat{H}^i(G,M)\otimes\widehat{H}^{-i-1}(G,\mathrm{Hom}(M,\mathbf{Q}/\mathbf{Z}))\longrightarrow\widehat{H}^{-1}(G,\mathbf{Q}/\mathbf{Z})$$

and

$$\widehat{H}^i(G,\mathbf{Z})\otimes\widehat{H}^{-i}(G,\mathbf{Z})\longrightarrow\widehat{H}^0(G,\mathbf{Z})$$

are duality pairings. This turns out to be a powerful tool to extend known results of ordinary cohomology to Tate cohomology.

A first result we get is that unlike in the ordinary cohomology case, $\widehat{H}^*(G,\mathbf{Z})$ is not noetherian unless G has periodic cohomology. Furthermore, all elements of negative degree are nilpotent. Thus, the product in negative degrees differs greatly from the product in positive degrees. This was already noted in [Ben–Car] for many cases for coefficients in a field of characteristic p.

We prove that the multiplicative structures of $\widehat{H}^*(G,\mathbf{Z})$ and $\widehat{H}^*(G,\mathbf{Z}/n\mathbf{Z})$ are uniquely identified by the additive structure of the Tate cohomology module, the cup product for the ordinary cohomology and a finite number of products in negative degree. When the ordinary cohomology ring is generated as a ring in degree 1 and 2, the knowledge of the cup product for ordinary cohomology is enough to determine all cup products for the Tate case. We apply this to abelian p-groups $\mathbf{Z}/p^a\mathbf{Z}\times\mathbf{Z}/p^b\mathbf{Z}$, getting the cup products explicitly. For groups $G=\mathbf{Z}/p\mathbf{Z}\times\mathbf{Z}/p\mathbf{Z}$, we also determine the structure of $\widehat{H}^*(G,\mathbf{Z}/p\mathbf{Z})$ both as a ring and as an $\widehat{H}^*(G,\mathbf{Z})$-algebra. In particular, we observe that $\widehat{H}^*(G,\mathbf{Z}/p\mathbf{Z})$ is finitely generated as a $\widehat{H}^*(G,\mathbf{Z})$-module.

General abelian p-groups have a set of additive generators of a particular form. This allows us to derive all the product structure apart from a finite number of couples of degrees. The products in negative degrees turn out to be zero apart from rare cases, which are *a priori* necessarily non-zero because of integral duality.

II.2 : Additive consequences of integral duality

Integral duality has interesting consequences for the additive and multiplicative structures of Tate cohomology rings and modules. As seen in Lemma I.4.1, integral duality implies that each element of the Tate cohomology ring with integer coefficients of order the order of the group is a unit in the ring. This implies the periodicity of the structure. If we consider the product of two homogeneous elements of the same degree, we get the following

Proposition II.2.1: *Let p be a prime and let p^{n_1} and p^{n_2} be the orders of two different cyclic direct summands of $\widehat{H}^i(G,\mathbf{Z})$ as an abelian group. Call p^{n_p} the biggest power of p dividing $\#G$. Then $n_1+n_2 \le n_p$. If i is odd and p is an odd prime, then both n_1 and n_2 are not greater than $n_p/2$.*

Proof: Suppose that the elements a_1 and a_2 lie in two different direct summands of $\widehat{H}^i(G,\mathbf{Z})$ and have orders p^{n_1} and p^{n_2} respectively, with $n_1+n_2 > n_p$. Then by integral duality there exist elements b_1 and b_2 in $\widehat{H}^{-i}(G,\mathbf{Z})$ such that

$$\begin{aligned} b_i a_i &= \frac{\#G}{\mathrm{ord}(a_i)} \in \widehat{H}^0(G,\mathbf{Z}) \text{ for } i \in \{1,2\} \\ b_i a_j &= 0 \qquad \text{if } i \neq j. \end{aligned}$$

Now, consider the product $b_1b_2a_1a_2$. Grouping it as $b_1a_1b_2a_2$, up to sign it equals

$$\frac{(\#G)^2}{\mathrm{ord}(a_1)\mathrm{ord}(a_2),}$$

which has order $p^{n_1+n_2}/(p^{n_p}) \ge p$ in $\widehat{H}^0(G,\mathbf{Z})$. But grouping it as $b_1a_2b_2a_1$, it equals 0 and we get the contradiction.

Let i be odd, let p be an odd prime and suppose there is an element a of order p^{n_1} greater than $\sqrt{p^{n_p}}$. Consider an element $b \in \widehat{H}^{-i}(G,\mathbf{Z})$ such that $ba = \#G/\mathrm{ord}(a)$. Then, as above,

$$abab = \frac{(\#G)^2}{(\mathrm{ord}(a))^2},$$

which is not 0. But by the anticommutativity of the cup product, as i is odd, $a^2 = -a^2$ must be zero, since its order is odd, and then so must be the product $abab$.

■

The result extends easily to elements of order not a prime power, taking into account the fact that the order of any element divides the cardinality of the group. We skip the proof of the following

Corollary II.2.2: *There can be at most one cyclic direct summand of $\widehat{H}^i(G, \mathbf{Z})$ of order greater than $\sqrt{\#G}$. If i and the order of G are odd, then all elements (and hence all cyclic summands) of $\widehat{H}^i(G, \mathbf{Z})$ have order at most $\sqrt{\#G}$.*

Summarizing, we have that one homogeneous element can have order at most the order of the group and that two linearly independent elements in the same homogeneous component can't have orders whose product exceeds the order of the group. One would be tempted to generalize the result to more than two elements. This is unfortunately impossible. For instance, if the group is the abelian p-group $(\mathbf{Z}/p\mathbf{Z})^2$, the cohomology modules $\widehat{H}^i(G, \mathbf{Z})$ are of the form $(\mathbf{Z}/p\mathbf{Z})^{n_i}$ with $n_i > 2$ as soon as i is greater than 5. Thus the orders of any three linearly independent homogeneous elements of the same high enough degree have a product exceeding p^2. This case will be studied in depth in later chapters.

II.3 : Multiplicative consequences of integral duality

From integral duality one can also deduce consequences on the multiplicative structure of Tate cohomology rings. It can be used to prove that, unlike in the case of ordinary cohomology rings, the Tate cohomology ring with integer coefficients is almost never noetherian. A case apart is when the cohomology is periodic (see Lemma I.4.1). In that case noetherianity is obvious. In fact if T is a period, then every ideal is the periodic repetition of a submodule of $\bigoplus_{0\leq i\leq T-1} \widehat{H}^i(G, \mathbf{Z})$, which is finite. Then every ascending chain is finite. We have the complete converse, i.e. in all other cases the ring $\widehat{H}^*(G, \mathbf{Z})$ is not noetherian. We prove a preliminary lemma:

Lemma II.3.1: *Let G be a non-trivial finite group. Then $H^*(G,\mathbf{Z})_p$ (and hence $\widehat{H}^*(G,\mathbf{Z})_p$) contains a non-nilpotent element of strictly positive degree for any prime p dividing the order of G.*

Proof: Take a cyclic p-subgroup $H \subset G$ and consider the coinduced module

$$M = \operatorname{Hom}_H(\mathbf{Z}[G], \mathbf{Z}).$$

Then by Shapiro's Theorem I.2.1 we have

$$H^*(G, M) \cong H^*(H, \mathbf{Z}).$$

Thus in even positive degree $H^*(G,M)$ is non-zero (and isomorphic to $\mathbf{Z}/p\mathbf{Z}$). By Proposition II.1.1, $H^*(G,M)$ is finitely generated over $H^*(G,\mathbf{Z})$ and the latter is a finitely generated ring. Then the submodule $\bigoplus_{i\geq 1} H^i(G,M)$ is finitely generated over $H^*(G,\mathbf{Z})_p$. Let's suppose all elements in $H^*(G,\mathbf{Z})_p$ to be nilpotent. It follows that $H^i(G,\mathbf{Z})_p = 0$ for $i \gg 1$. But then a graded module which is finitely generated over $H^*(G,\mathbf{Z})_p$ is also zero after a certain degree and we get a contradiction.

■

We now prove our theorem on noetherianity:

Theorem II.3.2: *Let G be a finite group with non-periodic cohomology. Then its Tate cohomology ring $\widehat{H}^*(G,\mathbf{Z})$ is not noetherian.*

Proof: By Lemma II.3.1, for any prime $p|\#G$ there is a non-nilpotent homogeneous element α_p of positive (even) degree in the p-part $\widehat{H}^*(G,\mathbf{Z})_p$. As G has non-periodic cohomology, there is one p for which G does not have p-periodic cohomology. Let α be a non-nilpotent homogeneous element in the corresponding p-part $\widehat{H}^*(G,\mathbf{Z})_p$ and consider the ideals

$$I_i := \operatorname{Ann}(\alpha^i) = \{x \in \widehat{H}^*(G,\mathbf{Z}) : x\alpha^i = 0\}$$

in $\widehat{H}^*(G,\mathbf{Z})$. Clearly we have $I_i \subset I_{i+1}$ for any $i \geq 0$. We are able to show that all inclusions are strict. In fact the order of α is a power of p between p and $\frac{\#G_p}{p}$. By integral duality, there is an element x_i in $\widehat{H}^{-\deg(\alpha^i)}(G,\mathbf{Z})$ whose cup product with α^i is $\frac{\#G}{\operatorname{ord}(\alpha^i)}$. Thus

$$\frac{\operatorname{ord}(\alpha^i)}{p} x_i \alpha^i = \frac{\#G}{p} \neq 0.$$

On the other hand,

$$\frac{\operatorname{ord}(\alpha^i)}{p} x_i {\alpha}^{i+1} = \frac{\#G}{p}\alpha = 0.$$

We have thus proved that the nonzero element $\frac{\operatorname{ord}(\alpha^i)}{p} x_i$ is in I_{i+1} but not in I_i. Hence $\widehat{H}^*(G, \mathbf{Z})$ is not noetherian.

■

One can actually prove that all homogeneous elements of negative degree in p-primary components for which the cohomology is not periodic are nilpotent.

Proposition II.3.3: *Suppose G is a finite group with non p-periodic cohomology for a p dividing its order. If $\widehat{H}^*(G, \mathbf{Z})_p$ is generated as a ring by elements $\alpha_1, \ldots, \alpha_n$ of degree $d_1, \ldots, d_n$ and the highest power of p dividing $\#G$ is p^l, then each homogeneous element of negative degree in $\widehat{H}^*(G, \mathbf{Z})_p$ is nilpotent. More precisely, its N-th power is zero, where $N = l \sum_{1 \le i \le n} d_i$.*

Proof: Let a be a homogeneous element in $\widehat{H}^{-d}(G, \mathbf{Z})_p$ with $d > 0$. Consider a^N, where N is the same as above. The degree of a^N is $-Nd$. We prove that it is zero proving that its product by a set of additive generators of $\widehat{H}^{dN}(G, \mathbf{Z})_p$ is zero. Consider an element $b = {\alpha_1}^{l_1} \ldots {\alpha_n}^{l_n}$ of degree dN. For at least one i we have $l_i \ge ld_i$. By hypothesis, the set of such elements generates the homogeneous component $\widehat{H}^{dN}(G, \mathbf{Z})_p$. It is possible to decompose $a^N b$ as a product of $a^{ld_i} \alpha_i^{ld}$ times a remaining factor. But up to sign, we have

$$a^{ld_i} \alpha_i^{ld} = (a^{d_i} \alpha_i^d)^l.$$

Since $a^{d_i} \alpha_i^d \in \widehat{H}^0(G, \mathbf{Z})_p$ is not a unit, it must be a multiple of p and so its l-th power is divided by p^l and hence is zero in $\widehat{H}^0(G, \mathbf{Z})_p$.

■

As any element in $\bigoplus_{i<0} \widehat{H}^i(G, \mathbf{Z})$ is sum of a finite number of homogeneous elements, the previous proposition implies the following corollary:

Corollary II.3.4: *Suppose G is a finite group with non p-periodic cohomology for a p dividing its order. Then any element $a \in \bigoplus_{i<0} \widehat{H}^i(G, \mathbf{Z})_p$ is nilpotent.*

We may attempt to extend these results in some ways to other modules of coefficients. This is particularly easy under the equivalent conditions of the following easy lemma:

Lemma II.3.5: *$\widehat{H}^*(G,M)$ contains a $\widehat{H}^*(G,\mathbf{Z})$-submodule isomorphic to $\widehat{H}^*(G,\mathbf{Z})_p$ if and only if it contains an element of order $\#G_p$.*

Proof: The "only if" part follows immediately from the fact that $\widehat{H}^0(G,\mathbf{Z})$ is cyclic of order $\#G$. For the "if" part, let's call t an element of order $\#G_p$. We can take it to be homogeneous. Then $\widehat{H}^*(G,\mathbf{Z})t$ is a free $\widehat{H}^*(G,\mathbf{Z})_p$-submodule of rank one. Indeed, by integral duality, for any homogeneous element $\alpha \in \widehat{H}^i(G,\mathbf{Z})_p$ there is an element $\widetilde{\alpha} \in \widehat{H}^{-i}(G,\mathbf{Z})$ such that $\widetilde{\alpha}\alpha = \dfrac{\#G}{\mathrm{ord}(\alpha)}$ in $\widehat{H}^0(G,\mathbf{Z})$. Then if α is non-zero, $\widetilde{\alpha}\alpha t \neq 0$ and hence $\alpha t \neq 0$. As this is true for any homogeneous $\alpha \in \widehat{H}^*(G,\mathbf{Z})_p \backslash \{0\}$ the lemma is proved.

■

Corollary II.3.6: *Let G be a finite group with non-p-periodic cohomology for a prime p dividing $\#G$ and let M be a G-module. If $\widehat{H}^*(G,M)$ contains an element t of order $\#G_p$, then $\widehat{H}^*(G,M)$ is not a noetherian $\widehat{H}^*(G,\mathbf{Z})$-module.*

Proof: By the previous lemma, the hypothesis implies the existence of a submodule isomorphic to $\widehat{H}^*(G,\mathbf{Z})_p$, which is not noetherian.

■

We also have

Corollary II.3.7: *If G doesn't have p-periodic cohomology for a prime $p|\#G$ and R is a commutative ring with trivial G-action such that $\widehat{H}^*(G,R)$ contains an element t of order $\#G_p$, then $\widehat{H}^*(G,R)$ is not noetherian.*

Proof: Let α and I_i be as in Theorem II.3.2. Then

$$\mathrm{Ann}(\alpha^i t) \cap \widehat{H}^*(G,\mathbf{Z})_p t = I_i t.$$

This is a strictly ascending chain of submodules and so $\{\mathrm{Ann}(\alpha^i t)\}$ is a strictly ascending chain of ideals.

■

II.4 : Multiplicative structure of Tate cohomology rings

The multiplicative structure of ordinary cohomology rings is known for a few cases. See for instance [Chap], where an explicit expression is given for abelian p-groups, from which the structure for all finite abelian groups is easily derived. See also [Ad–Mil], [Rus], [Car–3] for other examples.

Fewer examples appear in literature for the Tate case. If the finite group G has periodic cohomology, then the structure for positive degrees easily extends to all degrees. We recall from Lemma I.4.1 that in that case there exists a homogeneous unit u of strictly positive degree d in $\widehat{H}^*(G,\mathbf{Z})$. For any G-module M, cupping with u yields an isomorphism of degree d

$$\begin{aligned}\varphi_u : \widehat{H}^*(G,M) &\longrightarrow \widehat{H}^*(G,M)\\ x &\longmapsto ux\end{aligned}$$

Then every homogeneous element a has the form $u^i a'$ for an integer i and a homogeneous element a' of degree between 0 and $d-1$. If the module is a ring R with trivial G-action and $a = u^i a'$ and $b = u^j b'$ are two homogeneous elements in $\widehat{H}^*(G,R)$, by anticommutativity of the cup product one has

$$ab = u^i a' u^j b' = (-1)^{\deg(a')jd} u^i u^j a' b'$$

and thus up to sign ab is just a shift of $a'b'$ by a power of the isomorphism φ_u.

As an example, if G is cyclic it's a well-known fact that

$$\widehat{H}^*(G,\mathbf{Z}) \cong \mathbf{Z}/\#G\mathbf{Z}[T,T^{-1}]$$

with T homogeneous of degree 2.

In the sequel we show that integral duality is a powerful tool for extending results relative to ordinary cohomology to Tate cohomology. We are mainly interested in the cases of the ring of coefficients being $\mathbf{Z}$ or $\mathbf{Z}/n\mathbf{Z}$, with n a positive integer. Both cases satisfy the following hypothesis:

Hypothesis II.4.1: *There is an $r \in \mathbf{Z}$ such that $\widehat{H}^r(G,R)$ is cyclic and for any $i \in \mathbf{Z}$ the cup product*

$$\widehat{H}^i(G,R) \otimes \widehat{H}^{r-i}(G,R) \xrightarrow{\cup} \widehat{H}^r(G,R)$$

is a duality pairing.

The integer r will be 0 or -1 in the cases of interest.

We are able to prove that from the multiplicative structure for ordinary cohomology and the knowledge of the multiplication for a finite number of pairs of negative degrees one can derive the multiplicative structure for the entire Tate cohomology ring. The main result is the following:

Theorem II.4.2(Reduction Theorem): *Let $T = \bigoplus_{i\in\mathbf{Z}} T_i$ be a graded-commutative ring such that*

1. *T_i is a finitely generated module for all $i \in \mathbf{Z}$;*
2. *There exists an integer $r \leq 0$ such that T_r is cyclic and the products*

$$T_i \otimes T_{r-i} \longrightarrow T_r$$

   are duality pairings for all $i \in \mathbf{Z}$;

3. *The subring $\bigoplus_{i\geq 0} T_i$ is a finitely generated ring with generators of degree at most n.*

Let a graded-commutative ring $S = \bigoplus_{i\in\mathbf{Z}} S_i$ be given, together with an isomorphism of graded modules $\phi : S \longrightarrow T$ of degree 0, i.e. a family of isomorphisms of abelian groups

$$\phi_i : S_i \longrightarrow T_i,\ i \in \mathbf{Z}.$$

Suppose that $\phi(a)\phi(b) = \phi(ab)$ whenever the pair $(\deg(a), \deg(b))$ lies in one of the sets

$$\begin{aligned} K_1 &:= \{(i,j) \mid i+j = r\} \\ K_2 &:= \{(i,j) \mid i,j \geq 0\} \\ K_3 &:= \{(i,j) \mid r-n+1 \leq i,j \leq 0\}. \end{aligned}$$

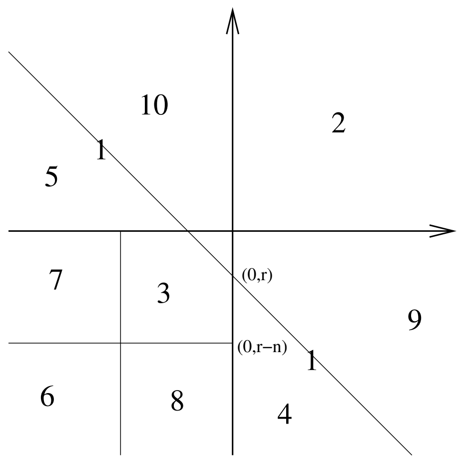
Then ϕ is an isomorphism of graded rings, i.e. $\phi(a)\phi(b) = \phi(ab)$ for all $a, b \in S$.

Observe that hypotesis 1 to 3 are also valid for S via the isomorphism ϕ^{-1}. In order to simplify the exposition, we introduce the set

$$K := \{(i,j) \mid i,j \in \mathbf{Z},\ \phi(a)\phi(b) = \phi(ab) \text{ for all } a\}.$$

As a visual aid, we draw a representation of the i, j plane divided into areas, corresponding to subsets $K_1, \ldots, K_{10}$ of K we will refer to in the course of the proof.

Diagram II.4.3



In the course of the proof we will make use of the following easy results:

Lemma II.4.4: *Under the hypothesis of Theorem II.4.2, if the pair (i, j) is in K, then also the pair (j, i) is.*

Proof: This follows from the graded-commutativity of S and T. Let a and b be elements of S_i and S_j respectively. If $\phi(a)\phi(b) = \phi(ab)$, then

$$\begin{aligned}\phi(ba) &= \phi((-1)^{\deg(a)\deg(b)}ab) \\ &= (-1)^{\deg(a)\deg(b)}\phi(ab) \\ &= (-1)^{\deg(a)\deg(b)}\phi(a)\phi(b) \\ &= \phi(b)\phi(a).\end{aligned}$$

■

Lemma II.4.5: *Under the hypothesis of Theorem II.4.2, the following are equivalent for homogeneous elements $a, b \in S_i$:*

$$\begin{gathered} a = b \\ ac = bc \quad \textit{for all } c \in S_{r-i} \\ \phi(a)\phi(c) = \phi(b)\phi(c) \quad \textit{for all } c \in S_{r-i} \end{gathered}$$

Proof: This is another formulation of hypothesis 2 of Theorem II.4.2.

■

Lemma II.4.6: *Under the hypothesis of Theorem II.4.2, if the pair (i, j) is in K, then also the pair $(i, r-i-j)$ is.*
Proof: Let a, b be elements of S_i and S_{r-i-j} respectively. By Lemma II.4.5,

$$\phi(a)\phi(b) = \phi(ab)$$

if and only if

$$\phi(a)\phi(b)\phi(c) = \phi(ab)\phi(c) \text{ for all } c \in S_j.$$

By hypothesis, $\phi(b)\phi(c) = \phi(bc)$ and we have

$$\phi(a)\phi(bc) = \phi(abc) = \phi(ab)\phi(c)$$

because the pairs $(\deg(a), \deg(bc))$ and $(\deg(ab), \deg(c))$ lie in K_1. The thesis follows.
■

Lemma II.4.7: *Let a be a graded-ring element of the form $a = \prod_{i=1}^{l} a_i$ with a_i homogeneous of degree $0 \leq d_i \leq n$. Let d' be an integer between 0 and $\deg(a)$. Then for any integer l' between 0 and l one has*

$$\max(0, d' - n + 1) \leq \prod_{i=0}^{l'} d_i \leq d'.$$

Proof: It is enough to take the partial product of highest degree not exceeding d'.
■

Proof of Theorem II.4.2: We will be done once we have proved that $K = \mathbf{Z} \times \mathbf{Z}$. By Lemma II.4.6 and Lemma II.4.4, K also contains the sets

$$K_4 := \{(i, j) \mid i \geq 0, j \leq r - i\}$$

and

$$K_5 := \{(i, j) \mid j \geq 0, i \leq r - j\} = \{(i, j) \mid (j, i) \in K_4\}$$

We now prove that

$$K_6 := \{(i, j) \mid i, j \leq r - n\}$$

is contained in K.
Let $a \in S_{-d_1}$ and $b \in S_{-d_2}$ with $d_1, d_2 \geq -r + n$. We want to show that

$$\phi(a)\phi(b)\phi(c) = \phi(ab)\phi(c)$$

for all c in a set of additive generators of $S_{d_1+d_2+r}$. Consider a homogeneous c of degree $d_1 + d_2 + r$ obtained as a product of homogeneous elements of $\bigoplus_{0\le i\le n} S_i$. By applying Lemma II.4.7 twice, we can group the factors into three elements c_1, c_2 and c_3 with $d_1+r-n < \deg(c_1) \le d_1+r$, $d_2+r-n < \deg(c_2) \le d_2+r$ and c_3 being the remaining factor. As the degrees of c_1, c_2 and c_3 are non-negative, $\phi(c)$ can be split as $\phi(c_1)\phi(c_2)\phi(c_3)$. The pairs $(d_1, \deg(c_1))$ and $(d_2, \deg(c_2))$ are in K_5, so

$$\phi(a)\phi(c_1) = \phi(ac_1)$$

and

$$\phi(b)\phi(c_2) = \phi(bc_2).$$

We also have

$$\phi(ac_1)\phi(bc_2) = \phi(ac_1bc_2),$$

as $(\deg(ac_1), \deg(bc_2))$ is in K_2. Finally, we also have

$$\phi(ac_1bc_2)\phi(c_3) = \phi(ac_1bc_2c_3),$$

because the pair of degrees lies in K_1, and

$$\phi(ab)\phi(c) = \phi(abc)$$

for the same reason.

The only remaining pairs of negative degrees are those in the set

$$K_7 := \{(i,j) \mid i \le r-n,\ r-n < j < 0\}$$

and its mirror image

$$K_8 := \{(i,j) \mid (j,i) \in K_7\}.$$

Consider a homogeneous element a of degree $-d_1 \le r-n$ and a homogeneous element b of negative degree $-d_2 < 0$. We need to show that

$$\phi(a)\phi(b)\phi(c) = \phi(ab)\phi(c)$$

for all c in a set of additive generators of $S_{d_1+d_2+r}$. Take c as the product of elements c_1 and c_2 with $d_1 - n + r < \deg(c_1) \le d_1 + r$. Such elements generate $S_{d_1+d_2+r}$ thanks to hypothesis 3 and Lemma II.4.7. Then $\phi(c) = \phi(c_1)\phi(c_2)$, $(d_1, \deg(c_1))$ is in K_5 and the degree of ac_1 is in the interval $(r-n, r]$.The pair $(\deg(ac_1), b)$ is in the set K_3, so we have

$$\phi(ac_1)\phi(b) = \phi(ac_1 b).$$

To conclude, $\deg(ac_1 b)$ and $\deg(c_2)$ have sum r, so

$$\phi(ac_1 b)\phi(c_2) = \phi(ac_1 bc_2).$$

From Lemma II.4.6, we finally extend to the remaining areas

$$K_9 := \{(i,j) \mid i \geq 0, -i+r < j < 0\}$$

and its mirror image

$$K_{10} := \{(i,j) \mid (j,i) \in K_9\}$$

and conclude the proof.

■

Theorem II.4.2 reduces the computation of the Tate cohomology ring of a finite group G to the computation of the ordinary cohomology ring plus the calculation of a finite number of products of cocycles of negative degree. It is possible to develop a computer program which computes the necessary products in negative-negative degrees. This will be the subject of Section III.

In some cases it is possible to extend the multiplicative structure for ordinary cohomology to Tate cohomology without the need to compute cup products in negative degree. For instance:

Proposition II.4.8: *Let G be a finite group such that the ordinary cohomology ring $H^*(G, \mathbf{Z})$ is generated by elements of degree at most 3. Let $S = \bigoplus_{i \in \mathbf{Z}} S_i$ be a graded-commutative ring and let $\phi : S \longrightarrow \widehat{H}^*(G, \mathbf{Z})$ be an isomorphism of graded modules. If $\phi(a)\phi(b) = \phi(ab)$ for every $a \in S_i$, $b \in S_j$ such that either $i, j \geq 0$ or $i = -j$, then ϕ is an isomorphism of graded rings.*

Proof: This follows from Theorem II.4.2, once we prove that $\phi(a)\phi(b) = \phi(ab)$ for all $a, b \in \bigoplus_{i \in \{1,2\}} S_i$. One needs to compute products

$$\widehat{H}^i(G, \mathbf{Z}) \otimes \widehat{H}^j(G, \mathbf{Z}) \longrightarrow \widehat{H}^{i+j}(G, \mathbf{Z})$$

for $i, j \in \{-1, -2\}$. Since $S_{-1} \cong \widehat{H}^{-1}(G, \mathbf{Z}) = \{0\}$, we only need to examine products in $S_{-2} \otimes S_{-2}$. If $a, b \in S_{-2}$, then

$$\phi(a)\phi(b) = \phi(ab)$$

if and only if

$$\phi(a)\phi(b)\phi(c) = \phi(ab)\phi(c)$$

for all c in a set additively generating S_4. Since both $\bigoplus_{i\geq 0} \widehat{H}^i$ and $\bigoplus_{i\geq 0} S_i$ are generated as rings by elements of degree 2 and 3, we can restrict to elements c of the form c_1c_2 with $c_1, c_2 \in S_2$. Now,

$$\begin{aligned}
\phi(a)\phi(b)\phi(c_1c_2) &= \phi(a)\phi(c_1)\phi(b)\phi(c_2) \\
&= \phi(ac_1)\phi(bc_2) \\
&= \phi(ac_1bc_2) \\
&= \phi(ab)\phi(c_1c_2),
\end{aligned}$$

so

$$\phi(a)\phi(b) = \phi(ab)$$

and the proof is complete.

■

A similar result is true for coefficients in a prime finite field:

Proposition II.4.9: *Let p be an odd prime and let G be a finite group with non p-periodic cohomology such that the ordinary cohomology ring $H^*(G, \mathbf{Z}/p\mathbf{Z})$ is generated by elements of degree at most 2. Let $S = \bigoplus_{i\in\mathbf{Z}} S_i$ be a graded-commutative ring and let $\phi : S \longrightarrow \widehat{H}^*(G, \mathbf{Z}/p\mathbf{Z})$ be an isomorphism of graded modules. Suppose that*

- *for every $a \in S_{-2}$ and $b \in S_2$ the product $ab \in S_0$ is zero;*
- *if $i, j \geq 0$ or $i + j = -1$ and $a \in S_i$, $b \in S_j$, then $\phi(a)\phi(b) = \phi(ab)$.*

Then ϕ is an isomorphism of graded rings.

Proof: As above, it is only necessary that products in S and $\widehat{H}^*(G, \mathbf{Z}/p\mathbf{Z})$ agree for negative degrees in the set $\{-1, -2\}$. The component of degree -1 is cyclic, thus by anticommutativity the product of two elements of degree -1 is 0 both in S and $\widehat{H}^*$. Since we are assuming that G doesn't have p-periodic cohomology, the products

$$\widehat{H}^i(G, \mathbf{Z}/p\mathbf{Z}) \otimes \widehat{H}^{-i}(G, \mathbf{Z}/p\mathbf{Z}) \longrightarrow \widehat{H}^0(G, \mathbf{Z}/p\mathbf{Z})$$

are zero for $i \neq 0$. Otherwise, the ring $\widehat{H}^*(G, \mathbf{Z}/p\mathbf{Z})$ contains a homogeneous unit of non-zero degree, so it is periodic. But from the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z} \xrightarrow{\pi} \mathbf{Z}/p\mathbf{Z} \longrightarrow 0$$

one easily derives that G must have periodic p-part cohomology, which contradicts the hypothesis.
Then, since $ab = 0 = \phi(a)\phi(b)$ for all $a \in S_{-2}$ and $b \in S_2$, by Lemma II.4.6 the products commute with ϕ for the pair of degrees $(-1,-2)$. If a and b have degree -2, instead, one has to study products abc for c varying in a set of additive generators of the degree 3 component. Such a set consists of products c_1c_2 with c_1 of degree 1 and c_2 of degree 2, as there is a set of multiplicative generators of $H^*(G, \mathbf{Z}/p\mathbf{Z})$ of degree 1 and 2. Now, $ac_2 = 0$ by hypothesis and $\phi(a)\phi(c_2) = 0$ by the reasoning above, so $ab = \phi(a)\phi(b) = 0$ and the hypothesis of Theorem II.4.2 are satisfied.

■

II.5 : Computation of $\widehat{H}^*(\mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}/p^b\mathbf{Z}, \mathbf{Z})$

The results of Chapter II.4 find a first application for finite abelian groups. In the abelian case, Tate and ordinary cohomology modules are the direct sum of the restrictions to Sylow subgroups, so it's enough to restrict the study to p-groups.

In [Chap], Chapman gives an explicit expression for the cup product for abelian p-groups of odd order ($p \neq 2$). He examines the cases with ring of coefficients $\mathbf{Z}$ or $\mathbf{Z}/p^l\mathbf{Z}$ where the exponent of G (i.e. the l.c.m. of the orders of the elements) divides p^l. Proposition II.4.8 applies in particular to the p-group $G := \mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}/p^b\mathbf{Z}$. We recall what Chapman's results ([Chap]) are in that case. We suppose $p \neq 2$ and $a \leq b$.

Proposition II.5.1(Chapman's results for $\mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}/p^b\mathbf{Z}$): *The ordinary cohomology ring $H^*(G, \mathbf{Z})$ is generated by elements α and β of degree 2 and γ of degree 3 with*

$$\mathrm{ord}(\alpha) = p^a$$
$$\mathrm{ord}(\beta) = p^b$$
$$\mathrm{ord}(\gamma) = p^a$$

only subject to the graded-commutative condition $\gamma^2 = 0$.

We have the following additive generators for the homogeneous components:

$$H^{2i}(G, \mathbf{Z}) \text{ is generated by } \{\alpha^j\beta^k \mid j, k \geq 0, j + k = i\}$$
$$H^{3+2i}(G, \mathbf{Z}) \text{ is generated by } \{\alpha^j\beta^k\gamma \mid j, k \geq 0, j + k = i\}$$

There is a natural notion of rank for an abelian p-group A. It can be defined as the dimension of A/pA over the finite field $\mathbf{F}_p$. Ranks and generating sets for the first homogeneous components of $H^*(G,\mathbf{Z})$ are resumed in the following table:

Table II.5.2

degree	**rank**	**generators**
0	1	1
1	0	
2	2	α, β
3	1	γ
4	3	$\alpha^2, \alpha\beta, \beta^2$
5	2	$\alpha\gamma, \beta\gamma$
6	4	$\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3$
7	3	$\alpha^2\gamma, \alpha\beta\gamma, \beta^2\gamma$
8	5	$\alpha^4, \alpha^3\beta, \alpha^2\beta^2, \alpha\beta^3, \beta^4$

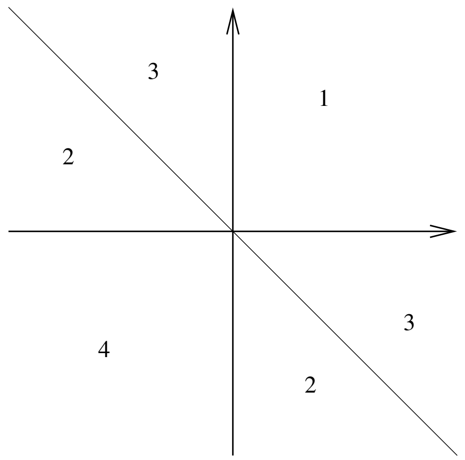
The order of an element $\alpha^i\beta^j\gamma^k$ is p^a if $i > 0$ or $k > 0$. The order of β^i is p^b if $i > 0$.

Corresponding to said generators, we have a “dual” basis for $\bigoplus_{i\leq 0}\widehat{H}^i(G,\mathbf{Z})$, which consists of the elements $D_A(\alpha^i\beta^j\gamma^k)$ for $i,j,k \geq 0$. This basis has the property that for elements $\alpha^i\beta^j\gamma^k$ and $\alpha^{i'}\beta^{j'}\gamma^{k'}$ *of the same degree*, one has

$$D_A(\alpha^i\beta^j\gamma^k)\alpha^{i'}\beta^{j'}\gamma^{k'} = \begin{cases} \dfrac{\#G}{\operatorname{ord}(\alpha^i\beta^j\gamma^k)} & \text{if } i=i',\ j=j' \text{ and } k=k' \\ 0 & \text{if } i\neq i' \text{ or } j\neq j' \text{ or } k\neq k' \end{cases}$$

As in Proposition II.4.8, the multiplicative structure extends uniquely to all degrees. The next theorem gives the structure of $\widehat{H}^*(G,\mathbf{Z})$. We express products according to numbered areas in the following diagram:

Diagram II.5.3



Theorem II.5.4: *Let G, α, β and γ be as above.*

1. *If $a = \alpha^{i_1}\beta^{j_1}\gamma^{k_1}$ and $b = \alpha^{i_2}\beta^{j_2}\gamma^{k_2}$ both have positive degree, then*

$$ab = \begin{cases} \alpha^{i_1+i_2}\beta^{j_1+j_2}\gamma^{k_1+k_2} & \text{if } k_1 + k_2 \leq 1 \\ 0 & \text{if } k_1 + k_2 \geq 2. \end{cases}$$

2. *If $a = D_A(\alpha^{i_1}\beta^{j_1}\gamma^{k_1})$ has negative degree and $b = \alpha^{i_2}\beta^{j_2}\gamma^{k_2}$ has positive degree with $\deg(a) \leq -\deg(b)$, that is the product has negative degree, then*

$$ab = \begin{cases} cD_A(\alpha^{i_1-i_2}\beta^{j_1-j_2}\gamma^{k_1-k_2}) & \text{if } i_1 \geq i_2,\ j_1 \geq j_2 \text{ and } k_1 \geq k_2; \\ 0 & \text{otherwise,} \end{cases}$$

where

$$c = \frac{\mathrm{ord}(\alpha^{i_1-i_2}\beta^{j_1-j_2}\gamma^{k_1-k_2})}{\mathrm{ord}(a)}.$$

3. *If $a = D_A(\alpha^{i_1}\beta^{j_1}\gamma^{k_1})$ has negative degree and $b = \alpha^{i_2}\beta^{j_2}\gamma^{k_2}$ has positive degree with $\deg(a) \geq -\deg(b)$, that is the product has positive degree, then*

$$ab = \begin{cases} p^a\beta^{j_2-j_1} & \text{if } i_1 = i_2 = 0,\ k_1 = k_2 = 0 \text{ and } j_2 \geq j_1 \\ 0 & \text{otherwise.} \end{cases}$$

4. *If $a = D_A(\alpha^{i_1}\beta^{j_1}\gamma^{k_1})$ and $b = D_A(\alpha^{i_2}\beta^{j_2}\gamma^{k_2})$ have negative degree, then*

$$ab = \begin{cases} p^a D_A(\beta^{j_1+j_2}) & \text{if } i_1 = i_2 = 0 \text{ and } k_1 = k_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Case 1 is just the product for ordinary cohomology, taking into account anticommutativity for γ.
In case 2, the only basis element of degree $-\deg(a)-\deg(b)$ whose product with b is not annihilated by a is the one whose exponents added with those in b add to i_1, j_1 and k_1 respectively, i.e. it is $c := \alpha^{i_1-i_2}\beta^{j_1-j_2}\gamma^{k_1-k_2}$. This is only possible under the conditions $i_1 \geq i_2$, $j_1 \geq j_2$ and $k_1 \geq k_2$. In this case ab is an integer multiple of $D_A(c)$. The factor arises from the fact that

$$abc = \frac{\#G}{\operatorname{ord}(bc)} = \frac{\#G}{\operatorname{ord}(a)}$$

while

$$D_A(c)c = \frac{\#G}{\operatorname{ord}(c)}.$$

Case 3 can be deduced from case 4 in view of Lemma II.4.5. For case 4, consider a basis element of degree $-\deg(a)-\deg(b) > 0$. It is a product of powers of α, β and γ. One can group factors into c_1 and c_2 so that $\deg(c_1) \in \{-\deg(a)-1, -\deg(a), -\deg(a)+1\}$ and $\deg(c_2) \in \{-\deg(b)-1, -\deg(b), -\deg(b)+1\}$. If ac_1 has degree ± 1, then it is zero. The only possibly non-zero product arises from c_1 and c_2 having degree $-\deg(a)$ and $-\deg(b)$ respectively. Now, $ac_1, bc_2 \in \widehat{H}^0(G, \mathbf{Z})$ have order at most the order of a and of b respectively, i.e.

$$ac_1bc_2 = k\frac{\#G}{\operatorname{ord}(a)}\frac{\#G}{\operatorname{ord}(b)}$$

for an integer k. Recalling that $\#G = 0$ in $\widehat{H}^0(G, \mathbf{Z})$ and that the order of a basis element can be either p^a or p^b, the only possibility for the product being non-zero is that both a and b have order p^b, i.e. that thay are of the form $D_A(\beta^{j_1})$ and $D_A(\beta^{j_2})$. In this case the only basis element c they possibly don't annihilate is $\beta^{j_1+j_2}$ and

$$ab\beta^{j_1+j_2} = a\beta^{j_1}b\beta^{j_2} = p^{2a} = p^a D_A(\beta^{j_1+j_2})\beta^{j_1+j_2}.$$

■

We now make some observations.

Observation II.5.5: *The ring $\widehat{H}^*(G, \mathbf{Z})$ is non-noetherian.*

This follows from Theorem II.3.2, since the group doesn't have periodic cohomology. Having an expression for the multiplicative structure, we can exhibit an ascending chain of ideals. Let's consider the ideals $(D_A(\alpha^i))$. One has

$$D_A(\alpha^i) = D_A(\alpha^{i+1})\alpha,$$

so there is an ascending chain

$$(D_A(\alpha)) \subset (D_A(\alpha^2)) \subset (D_A(\alpha^3)) \subset \dots$$

On the other hand, product of $D_A(\alpha^i)$ by an element of negative degree never yields the element $D_A(\alpha^{i+1})$, so the inclusions are strict.

Observation II.5.6: *All elements in $\bigoplus_{i<0} \widehat{H}^i(G, \mathbf{Z})$ are nilpotent.*

In fact, product of two elements of negative degree yields an element in

$$p^a \bigoplus_{i<0} \widehat{H}^i(G, \mathbf{Z}).$$

Iterating, the integer factor will eventually become a multiple of $p^{a+b} = \#G$.

Observation II.5.7: *All products in $\bigoplus_{i<0} \widehat{H}^i(G, \mathbf{Z})$ are zero if and only if $a = b$.*

If $a = b$, all products in case 4 of Theorem II.5.4 are zero. If $a < b$, instead,

$$D_A(\beta^i)D_A(\beta^j) = p^a D_A(\beta^{i+j})$$

is not zero, since the order of $D_A(\beta^{i+j})$ is $p^b > p^a$.

As to nilpotence, we notice that

Observation II.5.8: *For any integer $i > 0$ there is a finite group whose Tate cohomology ring contains a homogeneous element b of negative degree such that $b^{i-1} \neq 0$ but $b^i = 0$.*

Proof: It is enough to take the group $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p^i\mathbf{Z}$ and take as b the element $D_A(\beta)$, which has order p^i. One has that $D_A(\beta)^j = p^j D_A(\beta^j)$, so that $D_A(\beta)^j \neq 0$ for all $j < i$, while $D_A(\beta)^i = 0$.

■

II.6 : The structure of $\widehat{H}^*(\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$

If p is an odd prime, then Chapman ([Chap]) also gives the multiplicative structure for the ordinary cohomology ring of $\mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}/p^b\mathbf{Z}$ with coefficients in $\mathbf{Z}/p^c\mathbf{Z}$ with $c \geq a, b$. It is generated by elements of degree 1 and 2. In particular, we know it for $G = \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$ and in view of Proposition II.4.9, we can extend it to Tate cohomology.

Proposition II.6.1(Chapman for $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$): *The ordinary cohomology ring*

$$H^*(\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$$

is generated by elements x_1 and x_2 of degree 1 and y_1 and y_2 of degree 2, subject to the conditions

$$px_1 = px_2 = py_1 = py_2 = 0$$
$$x_1^2 = x_2^2 = 0$$

The homogeneous module $\widehat{H}^i(G, \mathbf{Z}/p\mathbf{Z})$ is generated by monomials in x_1, x_2, y_1 and y_2 with x_1 and x_2 appearing with exponent at most 1. For the first degrees, we write generators explicitly in the following table:

Table II.6.2

degree	**rank**	**generators**
0	1	1
1	2	x_1, x_2
2	3	x_1x_2, y_1, y_2
3	4	$x_1y_1, x_1y_2, x_2y_1, x_2y_2$
4	5	$x_1x_2y_1, x_1x_2y_2, y_1^2, y_1y_2, y_2^2$
5	6	$x_1y_1^2, x_1y_1y_2, x_1y_2^2, x_2y_1^2, x_2y_1y_2, x_2y_2^2$
6	7	$x_1x_2y_1^2, x_1x_2y_1y_2, x_1x_2y_2^2, y_1^3, y_1^2y_2, y_1y_2^2, y_2^3$

We call e a distinguished generator of $\widehat{H}^{-1}(G, \mathbf{Z}/p\mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z}$. For $i \geq 0$, $\widehat{H}^{-i-1}(G, \mathbf{Z}/p\mathbf{Z})$ has a basis dual to the basis of $\widehat{H}^i(G, \mathbf{Z}/p\mathbf{Z})$. If $a = x_1^{i_1} x_2^{j_1} y_1^{k_1} y_2^{l_1}$

and $b = x_1^{i_2} x_2^{j_2} y_1^{k_2} y_2^{l_2}$ are basis elements of $\widehat{H}^i(G, \mathbf{Z}/p\mathbf{Z})$, then the dual of a, which we denote by $D_B(a)$, is the homogeneous element in $\widehat{H}^{-i-1}(G, \mathbf{Z}/p\mathbf{Z})$ such that

$$D_B(a)b = \begin{cases} e & \text{if } i_1 = i_2, j_1 = j_2, k_1 = k_2, l_1 = l_2 \\ 0 & \text{otherwise} \end{cases}$$

To derive the product structure for all degrees, we make use of the following intermediate lemma:

Lemma II.6.3: *Let $G = \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$. Then the cup-products*

$$\widehat{H}^{-i}(G, \mathbf{Z}/p\mathbf{Z}) \otimes \widehat{H}^{-j}(G, \mathbf{Z}/p\mathbf{Z}) \longrightarrow \widehat{H}^{-i-j}(G, \mathbf{Z}/p\mathbf{Z})$$

are zero for all $i, j > 0$.

Proof: We call $B^i := \widehat{H}^i(G, \mathbf{Z}/p\mathbf{Z})$ for brevity. Let $a \in B^{-i}$ and $b \in B^{-j}$. Any element c in B^{i+j-1} of the form $x_1^{i_1} x_2^{i_2} y_1^{i_3} y_2^{i_4}$ can be written as product of two elements c_1 and c_2 of degrees i and $j-1$ or $i-1$ and j. Then, $abc = abc_1c_2 = \pm ac_1bc_2$. By the reasoning in the proof of Proposition II.4.9, either ac_1 or bc_2 are zero, since either $\deg(a) = -\deg(c_1)$ or $\deg(b) = -\deg(c_2)$. Then $abc = 0$ for all c in a generating set of B^{i+j-1}, then $ab = 0$.

■

The next proposition gives an expression for the cup products in $\widehat{H}^*(G, \mathbf{Z}/p\mathbf{Z})$:

Proposition II.6.4: *Let $G = \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$. Then the cup product in $\widehat{H}^*(G, \mathbf{Z}/p\mathbf{Z})$ satisfies the following conditions:*

1. *If $a = x_1^{i_1} x_2^{j_1} y_1^{k_1} y_2^{l_1}$ and $b = x_1^{i_2} x_2^{j_2} y_1^{k_2} y_2^{l_2}$ for $i_{1,2}, j_{1,2} \in \{0,1\}$ and $k_{1,2}, l_{1,2} \geq 0$ then*

$$ab = \begin{cases} 0 & \textit{if } i_1 = i_2 = 1 \textit{ or } j_1 = j_2 = 1 \\ -x_1^{i_1+i_2} x_2^{j_1+j_2} y_1^{k_1+k_2} y_2^{l_1+l_2} & \textit{if } i_1 = j_2 = 0 \textit{ and } i_2 = j_1 = 1 \\ x_1^{i_1+i_2} x_2^{j_1+j_2} y_1^{k_1+k_2} y_2^{l_1+l_2} & \textit{otherwise} \end{cases}$$

2. *If $a = D_B(x_1^{i_1} x_2^{j_1} y_1^{k_1} y_2^{l_1})$ of negative degree and $b = x_1^{i_2} x_2^{j_2} y_1^{k_2} y_2^{l_2}$ has positive degree with $\deg(a) + \deg(b) \leq -1$, then*

$$ab = \begin{cases} 0 & \textit{if } i_1 < i_2 \textit{ or } j_1 < j_2 \textit{ or } k_1 < k_2 \textit{ or } l_1 < l_2; \\ cD_B(x_1^{i_1-i_2} x_2^{j_1-j_2} y_1^{k_1-k_2} y_2^{l_1-l_2}) & \textit{otherwise,} \end{cases}$$

where

$$c = \begin{cases} -1 & \text{if } i_1 = j_1 = j_2 = 1 \text{ and } i_2 = 0; \\ 1 & \text{otherwise.} \end{cases}$$

3. *If* $\deg(a) < 0$ *and* $\deg(b) < 0$ *or* $\deg(a) > 0$ *and* $-\deg(a) - 1 < \deg(b) < 0$, *then* $ab = 0$.

Proof: The products for positive degree are simply those for ordinary cohomology. Those for case 2 are derived from case 1 by the reasoning in Lemma II.4.6. As to the rest, it is zero by Lemma II.6.3 and Lemma II.4.5.

■

The module $\widehat{H}^*(G, \mathbf{Z}/p\mathbf{Z})$ is also an example of $\widehat{H}^*(G, \mathbf{Z})$-module. It actually is a $\widehat{H}^*(G, \mathbf{Z})$-algebra. For the rest of this chapter, we will write A^i for $\widehat{H}^i(G, \mathbf{Z})$ and B^i for $\widehat{H}^i(G, \mathbf{Z}/p\mathbf{Z})$. It is possible to compute the image of the map

$$\begin{aligned} A^* &\longrightarrow B^* \\ a &\longmapsto a \cdot 1 \end{aligned}$$

We will need an explicit expression of the dimensions of A^i $(i \neq 0)$ and B^j as $\mathbf{Z}/p\mathbf{Z}$–vector spaces. From Table II.5.2 and Table II.6.2 one notices that increasing the degree by 2 also increases the rank by 2. From this, together with the two duality laws, one has

$$\begin{aligned} &\dim A^{2i} = \dim A^{-2i} = i + 1 \quad \text{if } i \neq 0 \\ &\dim A^{2j+1} = \dim A^{-2i-1} = i \\ &\dim B^i = \dim B^{-i-1} = i + 1 \end{aligned}$$

Furthermore, by the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z} \xrightarrow{\pi} \mathbf{Z}/p\mathbf{Z} \longrightarrow 0$$

we get the long exact sequence of cohomology

$$\cdots \xrightarrow{\delta} A^i \xrightarrow{p} A^i \xrightarrow{\pi^*} B^i \xrightarrow{\delta} A^{i+1} \xrightarrow{p} \cdots$$

As p annihilates A^i for any $i \neq 0$, for $i \neq -1, 0$ we have short exact sequences

$$0 \longrightarrow A^i \longrightarrow B^i \longrightarrow A^{i+1} \longrightarrow 0.$$

For the remaining indices we have

$$0 \longrightarrow B^{-1} \xrightarrow{\delta} A^0 \xrightarrow{p} A^0 \xrightarrow{\pi^*} B^0 \longrightarrow 0.$$

One may observe that these sequences are compatible with the count of dimensions.

We have the following result for the immersion of $\pi^*(A^*)$ in B^*:

Theorem II.6.5: *The image of $\bigoplus_{i\geq 0} A^i$ in B^* is generated (additively) by the elements*

$$y_1^i y_2^j (x_1y_2 - x_2y_1)^k$$

for $i, j \geq 0$ and $k \in \{0, 1\}$. For $n \geq 0$, the image of A^{-2n-1} in B is generated by the elements of the form

$$D_B(x_1x_2y_1^iy_2^{n-i})$$

for i running from 0 to n. The image of A^{-2n-2} is generated by the elements

$$D_B((x_1y_2 + x_2y_1)y_1^iy_2^{n-i-1})$$

with i running from 0 to $n-1$, plus the two elements $D_B(x_1y_1^n)$ and $D_B(x_2y_2^n)$.

Proof: We examine more carefully the behaviour of the map π^*. Note that the map $\delta : B^1 \longrightarrow A^2$ is surjective, and that any two generators of A^2 can be chosen as α and β. We choose $\alpha = \delta(x_1)$ and $\beta = \delta(x_2)$. Next we may take $y_1 = \pi^*(\alpha)$ and $y_2 = \pi^*(\beta)$. Because of the functoriality of the cup product, for $a \in A^i$, $b \in B^j$ we have $\delta(\pi^*(a)b) = a\delta b$. Applying this to $i = 2, j = 1$, we can calculate the kernel of $\delta : B^3 \longrightarrow A^4$, which is generated by $x_1y_2 - x_2y_1$. We may choose γ so that it's mapped by π^* to this element. As to negative degrees, we have $A^iA^{-i-1} \subset A^{-1} = 0$. So the image of A^{-i-1} in B must be ortogonal (with respect to the cup-product in B) to the image of A^i. From what we know on dimensions and from the fact that in negative degree π^* is an injection, we obtain that for $i \geq 0$ actually

$$\pi^*(A^{-i-1}) = \{b \in B^i \mid b \cup (\pi^*(A^i)) = 0\}.$$

It's just a matter of routine to check that the assertion follows.

■

It now follows easily that

Corollary II.6.6: *B is a finitely generated A–module.*

Proof: A set of generators is $\{1, x_1, x_2, x_1x_2\}$. They generate the positive part, as y_1, y_2 are in the image of A and x_1 and x_2 can only appear with exponent 0 or 1. For the negative part, we have that for any $i, j \geq 0$ the element $D_B(x_1x_2y_1^iy_2^j)$ is in the image of A. Multiplying it by x_1, x_2 or x_1x_2, in virtue of Proposition II.6.4 we obtain all elements of the form $D_B(x_1 y_1^i y_2^j)$, $D_B(x_2 y_1^i y_2^j)$ and $D_B(y_1^i y_2^j)$. These are enough to additively generate $\bigoplus_{i<0} B^i$.

■

II.7 : Partial results for abelian p-groups

For abelian p-groups of higher rank, one can exploit Chapman's results on generators of the ordinary cohomology ring with coefficients in $\mathbf{Z}$ to derive the Tate product structure almost everywhere.

Throughout this chapter, p denotes an odd prime.

Let G be the finite group $\prod_{i=1}^{l} \mathbf{Z}/p^{c_i}\mathbf{Z}$ with $c_1 \leq c_2 \leq \ldots c_l$. A set of multiplicative generators for $H^*(G, \mathbf{Z})$ must contain a set of additive generators of $H^2(G, \mathbf{Z})$, since $H^1(G, \mathbf{Z}) = 0$. For our case, a set of generators of $H^2(G, \mathbf{Z})$ is $\alpha_1, \ldots, \alpha_l$ with $\operatorname{ord}(\alpha_i) = p^{c_i}$. This set can be completed to a set of multiplicative generators of the whole ordinary cohomology ring $H^*(G, \mathbf{Z})$ by elements α_S with S a subset of $\{1, \ldots, l\}$ of cardinality at least 2. The degree of such an α_S equals $\#S + 1$ and its order is p^{c_k} where $k = \min(S)$. Elements α_i and α_S, corresponding to $\psi_{\{i\}}$ and ψ_S respectively in [Chap], are related to the cohomology of subgroups of G, but we will not exploit this fact here.

This is resumed in the following simplified version of Chapman's Proposition 7:

Proposition II.7.1: *The ordinary cohomology ring $H^*(G, \mathbf{Z})$ is additively generated by the set*

$$T := \left\{ \prod_{i=0}^{l} \alpha_i^{q_i} \right\} \bigcup \left\{ \alpha_S \prod_{i=0}^{l} \alpha_i^{q_i}, \#S \geq 2 \right\}$$

If one takes the product to start from $\min(s \in S)$, then the elements in the set generate independent cyclic summands of $H^*(G, \mathbf{Z})$.

As usual, the multiplicative structure for positive degrees extends to couples of degrees (i, j) with $i \geq 0$ and $j < -i$. The particular form for generators will enable us to derive the product structure for all couples of negative degrees $(-i, -j)$ with at least one between i and j exceeding l. We recognize the same behaviour of the case $\mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}/p^b\mathbf{Z}$, i.e. all products are 0 except those between elements of order exceeding $\sqrt{\#G}$ (in case they exist). The cup products for almost all negative degrees are given in the following theorem:

Theorem II.7.2: *Let $G = \prod_{i=1}^{l} \mathbf{Z}/p^{c_i}\mathbf{Z}$. For all $i > 0$, call $D_A(\alpha_l^i) \in \widehat{H}^{-2i}(G, \mathbf{Z})$ the dual of α_l^i according to a basis of $\widehat{H}^{2i}(G, \mathbf{Z})$ containing α_l^i. If a and b are homogeneous elements of negative degree of $\widehat{H}^*(G, \mathbf{Z})$, with $\deg(a) \leq -l$ and $\deg(b) < 0$, then*

$$ab = \begin{cases} 0 & \text{if } \operatorname{ord}(a)\operatorname{ord}(b) \leq \#G \\ \frac{\#G}{p^{c_l}} D_A(\alpha_l^{\frac{i+j}{2}}) & \text{if } i, j \text{ are even, } a = D_A(\alpha_l^{\frac{i}{2}}) \text{ and } b = D_A(\alpha_l^{\frac{j}{2}}). \end{cases}$$

In particular, if $p^{c_i} \leq \sqrt{\#G}$ for all i, then $ab = 0$ for all $a \in \bigoplus_{j \leq -l} \widehat{H}^j(G, \mathbf{Z})$, $b \in \bigoplus_{j<0} \widehat{H}^j(G, \mathbf{Z})$.

Proof: We first observe that the definition of $D_A(\alpha_l^i)$ depends on the choice of a basis, i.e. of a set generating a submodule of $\widehat{H}^{2i}(G, \mathbf{Z})$ complementary to $\langle \alpha_l^i \rangle$. The element $\frac{\#G}{p^{c_l}} D_A(\alpha_l^i)$, instead, is uniquely determined. In fact, two possible $D_A(\alpha_l^i)$'s differ by an element of order at most $p^{c_{l-1}}$, which is at most $\frac{\#G}{p^{c_l}}$. Then, the difference is annihilated by the integer factor.

As in Lemma II.4.5, if a and b are homogenous, then the computation of abc for all c in a generating set of $\widehat{H}^{-\deg(a)-\deg(b)}(G, \mathbf{Z})$ determines the product ab. Let $-i \leq -l$, $-j < 0$ be the degrees of a and b respectively. Any element c of degree $i + j$ in the generating set T introduced in Proposition II.7.1 can be split as product of two elements c_1 and c_2 of degrees either $i - 1$ and $j + 1$ or i and j or $i + 1$ and $j - 1$. Consider $abc = \pm(ac_1)(bc_2)$. As $\widehat{H}^1(G, \mathbf{Z}) = \widehat{H}^{-1}(G, \mathbf{Z}) = \{0\}$, it yields a possibly non-zero product only if

$$\deg(c_1) = i = -\deg(a)$$

and

$$\deg(c_2) = j = -\deg(b).$$

In this case, ac_1 is a multiple of $\frac{\#G}{\operatorname{ord}(a)}$ and bc_2 is a multiple of $\frac{\#G}{\operatorname{ord}(b)}$.

If $\operatorname{ord}(a)\operatorname{ord}(b) \leq \#G$, then the product ac_1bc_2 is a multiple of $\#G$, so it is zero in $\widehat{H}^0(G, \mathbf{Z})$. Hence the product ab is 0.

If $a = D_A(\alpha_l^{\frac{i}{2}})$ and $b = D_A(\alpha_l^{\frac{j}{2}})$, instead, one has

$$\begin{aligned} ab\alpha_l^{\frac{i+j}{2}} &= a\alpha_l^{\frac{i}{2}} b\alpha_l^{\frac{j}{2}} \\ &= \frac{(\#G)^2}{(\operatorname{ord}(\alpha_l))^2} \\ &= \frac{\#G}{\operatorname{ord}(\alpha_l)} D_A(\alpha_l^{\frac{i+j}{2}})\alpha_l^{\frac{i+j}{2}}, \end{aligned}$$

while $abc = 0$ for all other generators $c \in T$ of degree $i + j$.

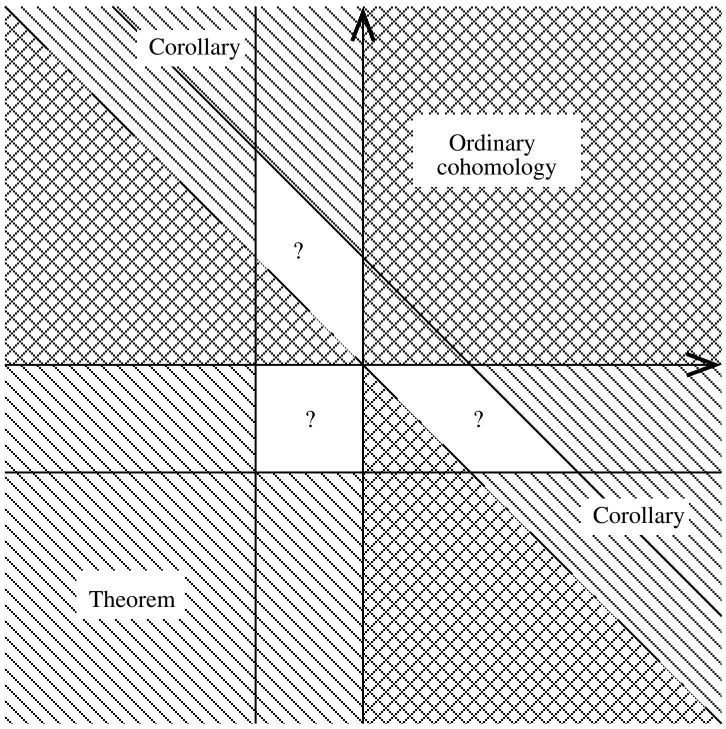
■

By applying Lemma II.4.5, one can deduce as a corollary to last theorem products for the following case:

Corollary II.7.3: *Under the same conditions as Theorem II.7.2, if a has negative degree $-i < 0$ and b has positive degree $j > 0$ such that either $j \geq i + l$ or $i \geq l$ and $j \geq i$, then*

$$ab = \begin{cases} 0 & \text{if } \operatorname{ord}(a)\operatorname{ord}(b) \leq \#G \\ \alpha_l^{\frac{j-i}{2}} & \text{if } i, j \text{ are even, } a = D_A(\alpha_l^{\frac{i}{2}}) \text{ and } b = \alpha_l^{\frac{j}{2}}. \end{cases}$$

The areas for which we know products are shown in the following diagram:
Diagram II.7.4



We also observe that in blank (unknown) areas, the couples (i, j) with $i = -1$ or $j = -1$ or $i + j = \pm 1$ give rise to zero products. Thus, we only left out products for a total of $3(l-2)^2$ couples of degrees. We would actually be able to derive all by graded-commutativity and by Lemma II.4.6 from the $(l-1)(l-2)/2$ couples with $-l < -i \leq -j < -1$.

II.8 : Further studies

Theorem II.7.2 does not complete the picture. One would like to extend the result to all couples of degrees, to get a complete Tate equivalent of Chapman's results of [Chap]. We are tempted to conclude in the most natural way with the following conjecture, extending Theorem II.7.2 to all negative degrees:

Conjecture II.8.1 (extension of Theorem II.7.2): *Let $G = \mathbf{Z}/p^{c_1}\mathbf{Z} \times \ldots \times \mathbf{Z}/p^{c_l}\mathbf{Z}$ for a prime p and for integers $1 \leq c_1 \leq \ldots \leq c_l$. Let α_l be an element of $\widehat{H}^2(G, \mathbf{Z})$ of order p^{c_l} and for $e > 0$ let $D_A(\alpha_l^e)$ be a dual of α_l^e with respect to a basis of $\widehat{H}^{2e}(G, \mathbf{Z})$ containing α_l^e. Let i and j be strictly positive and let a and b be elements of $\widehat{H}^*(G, \mathbf{Z})$ of degree $-i$ and $-j$ respectively. Then*

$$ab = \begin{cases} 0 & \textit{if } \operatorname{ord}(a)\operatorname{ord}(b) \leq \#G \\ \frac{p^{c_l}\#G}{\operatorname{ord}(a)\operatorname{ord}(b)} D_A(\alpha_l^{\frac{i+j}{2}}) & \textit{if } i, j \textit{ are even and } \operatorname{ord}(a)\operatorname{ord}(b) > \#G. \end{cases}$$

A consequence of this would be that for all abelian groups G the graded ring

$$\bigoplus_{i \leq 0} \widehat{H}^i(G, \mathbf{Z})$$

is commutative.

In [Ben–Car] the cup product for coefficients in $\mathbf{F}_p$ is studied and a few conditions are given that imply the nullity of cup products in negative degrees. In particular, for finite groups of the form $(\mathbf{Z}/p\mathbf{Z})^l$ with p a prime (not necessarily odd) and $l \geq 2$, Theorem 3.3 in [Ben–Car] implies that cup products

$$\widehat{H}^i(G, \mathbf{F}_p) \otimes \widehat{H}^j(G, \mathbf{F}_p) \longrightarrow \widehat{H}^{i+j}(G, \mathbf{F}_p)$$

are zero whenever i and j are negative. This implies that the groups $(\mathbf{Z}/p\mathbf{Z})^l$ satisfy the Conjecture II.8.1. In fact, the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z} \xrightarrow{\pi} \mathbf{F}_p \longrightarrow 0$$

determines injections

$$\widehat{H}^i(G, \mathbf{Z}) \hookrightarrow \widehat{H}^i(G, \mathbf{F}_p)$$

for $i < 0$. The injections commute with the cup product, so all cup products in negative degree are zero and the Conjecture II.8.1 is satisfied.

III : Computation of Tate cohomology

Introduction

Apart from a few cases, computing Tate cohomology modules and especially cup-products by pen and paper is a rather hard task. On the other hand, the subject has a very combinatorial flavour which makes it appealing to think of a computer implementation: groups are finite, modules are finitely generated and one has explicit recipes to create free resolutions. The temptation arises to feed the data to a calculator and let it do the hard work. Some work has already been done in this direction, leading to results such as [Rus], [Car–3], [Car–1], [Car–2] and [Whi]. These computations were carried out in specific settings suitable for strong optimization. For instance, choosing $\mathbf{Z}/2\mathbf{Z}$ as ring of coefficients allows one to store all coefficients as single bits and avoids coefficient-explosion phenomena, while saving space considerably.

We implemented a C++ library, following as closely as possible the logic hyerarchy of the algebraic objects of interest. In the end we had functions allowing the computation of Tate cohomology modules and cup products for arbitrary finite groups and modules, plus a specialized class for the computation of the cup product with coefficients in $\mathbf{Z}$. While theoretically working for arbitrary finite groups, performance limitations only let us carry out computations for small abelian groups.

In this section we point out some of the problems one encounters in such projects and show some of the choices we made. Finally, we show some sample runs, which confirm the theoretical results of Section II.

The source code developed is available for download at the address

`http://www.beautylabs.net/software/cohomology.html`.

We hope that someone will find it useful and that future improvements will put more complex computations within reach.

III.1 : Finite groups

The computational theory of finite groups is a widely studied subject. In our work, however, we didn't make use of deep results, being mainly interested in abelian groups.

To store a general finite groups, one can make use of a *matrix representation*. If elements are called $g_0, \ldots, g_{n-1}$, with n being the cardinality of the group, then the place $M_{i,j}$ of the structure matrix M contains the number l such that $g_l = g_i g_j$. As examples, the permutation group over 3 elements S_3 has the structure matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 4 & 5 & 2 & 3 \\ 2 & 5 & 0 & 4 & 3 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 3 & 1 & 2 & 5 & 0 \\ 5 & 2 & 3 & 1 & 0 & 4 \end{pmatrix}$$

where group elements correspond to the following permutations:

$$\begin{aligned} g_0 &\longleftrightarrow () \\ g_1 &\longleftrightarrow (23) \\ g_2 &\longleftrightarrow (13) \\ g_3 &\longleftrightarrow (12) \\ g_4 &\longleftrightarrow (123) \\ g_5 &\longleftrightarrow (132) \end{aligned}$$

The group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$, instead, has the following structure matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 3 & 2 & 5 & 4 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 2 & 5 & 4 & 1 & 0 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 4 & 1 & 0 & 3 & 2 \end{pmatrix}$$

considering the elements ordered as $(0,0),(1,0),(0,1),(1,1),(0,2),(1,2)$. Each line and column is a permutation of the set $\{0,\ldots,n-1\}$ and no two lines or two columns are the same. The line and column corresponding to the identity element contain the elements in the natural order.

This representation is overkill for abelian groups, which are better represented as a product of cyclic groups by an array of integers. For instance, the array representation of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ is $[2,3]$. To decrease complexity when working with abelian groups, while keeping able to deal with general finite groups, we used a mixed representation. In our implementation a group is represented as the direct product of an abelian and a generic component. A group element is represented as $[a_1,\ldots,a_n;l]$, with the array $[a_1,\ldots,a_n]$ containing the coefficients in the cyclic components of the abelian part and l being the index of the generic part, with the convention that the 0-th element is the identity. We note that the object oriented approach makes the substitution of more sophisticated implementations very easy, provided the interface is kept the same.

In our case, apart from standard group operations we need a set of generators of the group and a function expressing any element in terms of the generators. This is used for instance to compute the action of any element on a module when only the actions of the generators are stored.

III.2 : Z-modules

To deal with finitely generated $\mathbf{Z}$-modules, one needs the machinery of linear algebra over the integers. This differs in various ways from linear algebra over a field. For instance, from a set of vectors generating a lattice of dimension n one can not necessarily extract a generating subset with n elements. Take for instance the case $v_1 = (2)$, $v_2 = (3)$. Their $\mathbf{Z}$-span is $\mathbf{Z}$, but neither v_1 nor v_2 generate $\mathbf{Z}$.

All matrix elements from now on will be integers unless stated otherwise. For an introduction to the computational aspects of linear algebra over the integers, take a look at [Coh]. All modules in this section will be finitely generated.

A (finitely generated) free $\mathbf{Z}$-module is isomorphic to the direct sum of a finite number of copies of $\mathbf{Z}$, so it may just be identified by a non-negative integer. Apart from the isomorphism class, one often needs also the immersion in a given module. This can be either achieved by storing the immersion map separately or by keeping track of it in the module itself, by storing the corresponding matrix.

We introduce the following notation: if M is a matrix with n rows and m columns, the submodule of $\mathbf{Z}^n$ generated by the m column vectors of M is denoted by $\langle M\rangle$. A (finitely-generated) $\mathbf{Z}$-module is the quotient of a free module by a submodule. Each $\mathbf{Z}$-module is represented by two matrices M_G and M_R. It corresponds to the quotient $\langle M_G\rangle/\langle M_G M_R\rangle$. Some relations may be implicit in the generator matrix when the rank of M_G is less than its number of columns. In this case, we suppose that M_R takes these relations into account, i.e. that the kernel of M_G is in the image of M_R. This way, M_R alone determines the isomorphism class of the module.

For instance, the module A generated by matrix

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

with relation matrix (over the generators)

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

may be stored as the couple of matrices M and M_R with

$$M_R = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$$

since

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

generates the kernel of M.

It is advisable to treat the cases with a null relation matrix (free $\mathbf{Z}$-modules) or generated by the identity matrix (standard generators) in a special way. This allows to skip some computations in a few cases.

Recall that an $m \times n$ matrix S is in *Smith Normal Form* if

- $S_{i,j} = 0$ for all $i \neq j$;
- $S_{i,i} \geq 0$ for all $i < \min(m, n)$;
- $S_{i+1,i+1}$ divides $S_{i,i}$ for all $0 \leq i < \mathrm{rnk}(S)$;
- $S_{i,i} = 0$ for all $i \geq \mathrm{rnk}(S)$.

Given any integer $m \times n$ matrix M there is a unique $m \times n$ matrix in Smith Normal Form of the form

$$S = UMV$$

for suitable invertible integer matrices U and V. The Smith Normal Form of the relation matrix yields the structure of the module as product of cyclic groups. The finite elements on the diagonal of the SNF of an $n \times m$ matrix M are the orders of the cyclic components of the module

$$\frac{\mathbf{Z}^n}{\langle M \rangle.}$$

The zero elements correspond to infinite cyclic components.

If M and N are modules with generators $v_0, \ldots, v_{m-1}$ and $w_0, \ldots, w_{n-1}$ respectively, then $M \otimes N$ is a quotient of $\mathbf{Z}^{mn}$, where for $0 \leq i < m$ and $0 \leq j < n$, generator $ni + j$ corresponds to $v_i \otimes w_j$. The module $\mathrm{Hom}(M, N)$ is represented as a subquotient of $\mathbf{Z}^{mn}$, with the convention that for $0 \leq i < m$ and $0 \leq j < n$, the base element $ni+j$ of $\mathbf{Z}^{mn}$ corresponds to the map sending v_i to w_j and all other v_k's to 0.

A map of $\mathbf{Z}$-modules is just represented by an $n \times m$-matrix, where n and m are the number of generators of the codomain and domain respectively. The coefficients of the matrix can be reduced modulo the exponent of the codomain, in case it is finite. The image of a map is the submodule of the codomain generated (in terms of its generators) by the columns of the matrix corresponding to the map. To compute the kernel of a map, one has to compute the inverse image of the relation matrix of the codomain.

III.3 : G-modules

A G-module is a $\mathbf{Z}$-module with an action of G. For every element of the finite group G, one has a bijective $\mathbf{Z}$-map from the module to itself. One can just store the action matrices for elements in a standard set $\{g_i\}$ of generators of the group and compute the other ones on the fly. To exploit this, one needs a function returning, for every group element h, its expression in terms of the g_i's. The function may return an array of couples of values l_i, e_i such that

$$h = \prod_i g_{l_i}^{e_i}.$$

Then, if M_i is the matrix corresponding to generator g_i, the action of element h is given by the matrix $\prod_i M_{l_i}^{e_i}$. Since matrix inversion is much more expensive than the product, one reduces each e_i modulo $\mathrm{ord}(g_i)$ so that exponents are all positive and as small as possible. Exponentiation of matrices can make use of one of the general purpose binary exponentiation algorithms.

As to maps of G-modules, they are $\mathbf{Z}$-maps compatible with the action of G on domain and codomain. This condition makes the usual representation somehow redundant. In fact, by the compatibility and linearity conditions one has

$$f(\sum_i c_i g_i a_i) = \sum_i c_i g_i f(a_i)$$

for all $c_i \in \mathbf{Z}$, $g_i \in G$ and a_i in the domain. This implies that the map is uniquely determined by its values on a set $\{a_1, \ldots, a_l\}$ generating the domain over $\mathbf{Z}[G]$. In order to efficiently exploit this, one would need a way to

- identify a (possibly minimal) generating set $\{a_1, \ldots, a_l\}$;
- decompose any element of the domain in the form $\sum_i c_i g_i a_i$.

This introduces new issues and adds computational complexity to the problem. In the case of $\mathbf{Z}[G]$-free domains, though, the situation is particularly simple and worth considering, since such objects appear in free resolutions and in the computation of cup products. Let N be a G-module. The map $\mathbf{Z}^i \longrightarrow \mathbf{Z}[G]^i$ sending $(1, 0, \ldots, 0)$ to $((e), 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$ to $(0, (e), 0, \ldots, 0)$ and so on induces an isomorphism

$$\mathrm{Hom}_G(\mathbf{Z}[G]^i, N) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^i, N).$$

Thus, in storing a G-map whose domain is $\mathbf{Z}[G]$-free, it is possible to save space by only storing the columns of index divided by $\#G$. The inverse application, needed to reconstruct the original map, sends a map $f : \mathbf{Z}^i \longrightarrow N$ to the map $\widehat{f}$ such that

$$\begin{aligned}
\widehat{f}((g), 0, \ldots, 0) &= gf(1, 0, \ldots, 0) \\
\widehat{f}(0, (g), 0, \ldots, 0) &= gf(0, 1, 0, \ldots, 0) \\
\widehat{f}(0, 0, (g), 0, \ldots, 0) &= gf(0, 0, 1, 0, \ldots, 0) \\
&\cdots
\end{aligned}$$

We say that $\widehat{f}$ is the *G-extension* of f.

III.4 : Free resolutions

In order to compute Tate cohomology modules, we need to compute complete free resolutions. Even if appealing on a theoretical point of view for its simplicity, the standard resolution described in Chapter I.2 is computationally very expensive. In fact, Sr_i is isomorphic to the product of $(\#G)^i$ copies of $\mathbf{Z}[G]$ and its $\mathbf{Z}$-rank is $(\#G)^{i+1}$. The analysis of the case of a cyclic group shows that, at least in some cases, it is possible to do much better. For a cyclic group, a free resolution is given by

$$\ldots \xrightarrow{(g)-(e)} \mathbf{Z}[g] \xrightarrow{N} \mathbf{Z}[g] \xrightarrow{(g)-(e)} \mathbf{Z}[g] \xrightarrow{N} \mathbf{Z}[g] \xrightarrow{(g)-(e)} \mathbf{Z}[g] \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0.$$

Here N is multiplication by the norm element $\sum_{h \in G}(h)$ and g is a generator of G.

It is possible to figure out a lower limit for the complexity of a free resolution P_*. The cohomology module $H^i(G, M)$ is a subquotient of $\mathrm{Hom}_G(P_i, M)$. If P_i has rank r_i as a free $\mathbf{Z}[G]$-module, then $\mathrm{Hom}_G(P_i, M)$ is isomorphic as a $\mathbf{Z}$-module to $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^{r_i}, M)$, as described in Chapter III.3. Then its rank is r_i times the rank of M. It follows that any subquotient, hence also the cohomology module $H^i(G, M)$, has rank at most r_i times the rank of M. Thus the G-rank of P_i is at least

$$\mathrm{rnk}(H^i(G, M))/\mathrm{rnk}(M)$$

for all G-modules M. As explained in Theorem 7.4 of [Ad–Mil], one actually has the following converse:

Theorem III.4.1: *Let G be a finite p-group. A projective resolution $\varepsilon : P \longrightarrow \mathbf{Z}$ is minimal if and only if*

$$\mathrm{rnk}(P_i) = \#G \cdot \dim\left(H^i(G, \mathbf{F}_p)\right).$$

We now show how to get a minimal resolution Mr_*. This construction is referred to in the course of the proof of Theorem 7.4 of [Ad–Mil]. Suppose one has already constructed

$$\mathrm{Mr}_i \xrightarrow{d_i} \mathrm{Mr}_{i-1} \xrightarrow{d_{i-1}} \dots \xrightarrow{d_1} \mathrm{Mr}_0 \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0.$$

One has to find a minimal set of $\mathbf{Z}[G]$-generators of $\mathrm{Ker}(d_i)$, i.e. a minimal set of elements $a_1, \dots, a_n \in \mathrm{Ker}(d_i)$ such that

$$\sum_{j=1}^{n} \mathbf{Z}[G]a_j = \mathrm{Ker}(d_i).$$

Then one takes Mr_{i+1} to be $\mathbf{Z}[G]^n$ and the boundary map d_{i+1} is the G-extension of the $\mathbf{Z}$-map from $\mathbf{Z}^n$ to Mr_i sending the j-th standard basis element to a_j.

To find a minimal set of $\mathbf{Z}[G]$-generators of a module M, one can look at its module of coinvariants

$$M_G = \frac{M}{\sum_{g\in G}[(g)-(e)]M}.$$

The G action on M_G is trivial and any set of generators of M_G when lifted to M generate it as a G-module. One can also observe that if S is a set of generators of G, then

$$\sum_{g\in G}[(g)-(e)]M = \sum_{g\in S}[(g)-(e)]M,$$

This follows by the observation that, given two group elements g and h and an element $m \in M$, one has

$$[(gh)-(e)](m) = [(g)-(e)](h(m)) + [(h)-(e)](m).$$

To find generators of the quotient of a free module by a (necessarily free) submodule, one can make use of the Smith Normal Form. Suppose a submodule of $\mathbf{Z}^n$ is generated by the column vectors of matrix A, an $n \times r$ matrix of rank r. One can revert to this case by taking a generating matrix of a submodule by means of an LLL reduction algorithm or otherwise. Suppose $S = UAV$ is the Smith Normal Form of A, with U and V invertible. Then $AV = U^{-1}S$. One has

$$\langle A\rangle = \langle AV\rangle = \langle U^{-1}S\rangle$$

and

$$\mathbf{Z}^n = \langle U^{-1}\rangle.$$

If S has the form

$$\begin{pmatrix} \mathrm{Id}_l & 0 \\ 0 & D \end{pmatrix}$$

where Id_l is the $l \times l$ identity matrix and D is an $n-1 \times n-1$ diagonal matrix, then the first l columns of U^{-1} are in $\langle A\rangle$ and the columns after the l-th one constitute a minimal set completing $\langle A\rangle$ to all of $\mathbf{Z}^n$. If readily available SNF algorithms don't simultaneously compute U^{-1} (as is the case with LiDIA), the inversion of U can be time consuming.

One actually needs something weaker than an inverse of U. In fact,

$$\mathbf{Z}^n = \langle A\rangle + \langle T\rangle$$

if and only if

$$\mathbf{Z}^n = \langle UA\rangle + \langle UT\rangle.$$

Now,

$$\langle UA\rangle = \langle UAV\rangle = \langle S\rangle,$$

so any minimal complement of $\langle UA\rangle$ has the form

$$\left\langle \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \right\rangle,$$

with T_1 any $l \times (n-l)$ matrix and T_2 invertible of size $(n-l) \times (n-l)$. Calling U_2 the submatrix of U obtained by the last $n-l$ rows, UT has the form

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

as above if and only if U_2T is invertible of size $(n-l) \times (n-l)$. Such a T is provided by any algorithm computing the minimal integer image of a matrix (U_2 in this case), while computing at the same time the transformation matrix (our T). Such algorithms are provided in LiDIA for the computation of the LLL reduced form.

We resume the process in algorithm form:

Algorithm III.4.2:

Input: An $n \times m$ matrix A

Output: A minimal set completing the columns of A to a set generating $\mathbf{Z}^n$

- Transform A to LLL-reduced form, skipping zero columns, so that the resulting $n \times r$ matrix has rank r;
- Compute $S \leftarrow \mathrm{SNF}(A) = UAV$ an $n \times r$ matrix;
- Set $t = \max\bigl(\{i \mid U_{i,i} = 1\}\bigr)$;
- Set $U_2 = \begin{pmatrix} u_{1,t+1} & \cdots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{n,t+1} & \cdots & u_{n,n} \end{pmatrix}$
- Compute $\mathrm{LLL}(U_2) = U_2 T$; T is the $n \times n$ transformation matrix;
- Set $T_1 = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n-t} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \cdots & T_{n,n-t} \end{pmatrix}$
- Return T_1, possibly after LLL reduction.

To perform computations on Tate cohomology, one needs a complete resolution. We do this by "joining" together the positive resolution and its dual. For $i \geq 0$, the module P_{-i-1} is $\mathrm{Hom}(P_i, \mathbf{Z})$ with the G-action

$$g(f) = f \circ g^{-1}.$$

It is isomorphic to P_i. Choosing a $\mathbf{Z}$-basis for P_i invariant under the action of G, the corresponding duality isomorphism $a \longmapsto \widehat{a}$ from P_i to P_{-i-1} is a G-map. In fact, for a group element g and an element $a \in P_i$, one has

$$\widehat{ga} = \widehat{a} \circ g^{-1} = g\widehat{a}.$$

The boundary map d_{-i} is the image of d_i via the functor $\mathrm{Hom}(\,\_\,, \mathbf{Z})$. Its computation is very efficient, consisting of just a matrix transposition. The matrix representing the map $d_0 : P_0 \longrightarrow P_{-1}$ is square with all entries set as 1.

Once one has a complete free resolution P_*, computation of the cohomology modules requires computing the cochain complex $\mathrm{Hom}_G(P_*, M)$ and computing its cohomology.

The computation of $\mathrm{Hom}_G(P_i, M)$ is straightforward. As P_i is a free G-module, $\mathrm{Hom}_G(P_i, M) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^r, M)$ with r the G-rank of P_i. The computation of the cohomology modules requires the standard computation of kernel and image of a G-map and the quotient of modules. The G-structure on the modules can be forgotten while computing the cohomology modules, because the induced action of G is trivial.

III.5 : Cup product

In literature a few ways are described to compute cup products. In [Cas–Fro] an explicit formula is given for the standard resolution. The general theory, as developed, for instance, in [Br]or [C–E], makes use of a diagonal approximation on the complete resolution. This is a collection of maps

$$\Phi_{i,j} : P_{i+j} \longrightarrow P_i \otimes P_j$$

satisfying certain commutativity relations with respect to the boundary and augmentation maps of P (cfr. Chapter XII.4 in [C–E]). This is the hard step. Once one has defined these maps, if the homogeneous elements $a \in \widehat{H}^i(G, M)$ and $b \in \widehat{H}^j(G, N)$ are represented by maps

$$a : P_i \longrightarrow M$$
$$b : P_j \longrightarrow N,$$

the cup product ab is represented by $(a \otimes b) \circ \Phi_{i,j}$. Thus, most of the work is independent of the modules and could be carried out once and for all for any given group. A "recipe" for the construction of the diagonal approximation is given in Chapter XII.4 of [C–E] and we chose to follow this. The construction is of inductive nature. Required elements are:

- a $\mathbf{Z}$-endomorphism ρ of degree 0 of P such that $N\rho = \mathrm{Id}$, where N denotes the norm element, sending as usual a function f to

$$\sum_{g \in G} g \circ f \circ g^{-1}.$$

- a contracting homotopy s, i.e. a collection of $\mathbf{Z}$-maps $s_i : P_i \longrightarrow P_{i+1}$ such that $s_{i-1} \circ d_i + d_{i+1} \circ s_i = \mathrm{Id}_{P_i}$;

A morphism ρ is easily computed. If P_i is $\mathbf{Z}[G]^r$, then

$$\rho((g_1), \ldots, (g_r)) = \begin{cases} ((g_1), \ldots, (g_r)) & \text{if } g_1 = e \\ 0 & \text{if } g_1 \neq e \end{cases}$$

As to the contracting homotopy, it is easily derived from the decomposition of P_i as direct sum $K_i \oplus C_i$, where $K_i = \mathrm{Ker}(d_i)$ and C_i is any complement to K_i in P_i. Let k_i and c_i be the projections of P_i on K_i and C_i respectively. Then the contracting homotopy s_i is the only map such that the following diagram commutes:

$$\begin{array}{ccc} P_{i+1} & \xrightarrow{d_{i+1}} & P_i \\ k_{i+1}\downarrow & & \downarrow c_i \\ P_{i+1} & \xleftarrow{s_i} & P_i \end{array}$$

The inductive construction of the diagonal approximation proceeds in the following way:

$$\begin{aligned} \Phi_{0,0} : P_0 &\longrightarrow P_0 \otimes P_0 \\ (g) &\longmapsto (g) \otimes (g) \end{aligned}$$

For $i > 0$,

$$\begin{aligned} \Phi_{i,-i} &= -N(\rho'' s' d'' \Phi_{i-1,1-i}) \\ \Phi_{-i,i} &= -N(\rho' s'' d' \Phi_{i-1,1-i}) \end{aligned}$$

where for a map f, f' denotes $f \otimes \mathrm{Id}$ and $f'' = \mathrm{Id} \otimes f$, with the convention that for two maps f and g of degree d_1 and d_2 respectively, one has

$$(f \otimes g)(a \otimes b) = (-1)^{d_2 \deg(a)} f(a) \otimes g(b).$$

If $i + j < 0$, then

$$\Phi_{i,j} = N(\rho' \Psi_{i,j} s)$$

where

$$\Psi_{i,j} = d' \Phi_{i+1,j} + d'' \Phi_{i,j+1}.$$

If $i + j > 0$, then

$$\Phi_{i,j} = N(s' \Phi_{i-1,j} d\rho).$$

If the i-th module P_i of a resolution has rank r_i, then the maps involved, when considered as matrices, have the following dimensions:

Table III.5.1

Map	rows	columns	
$\Phi_{i,j}$	$r_i r_j$	r_{i+j}	
ρ_i	r_i	r_i	
d_i	r_{i-1}	r_i	
$f \otimes \mathrm{Id}_j$	nr_j	mr_j	if f has dimension $n \times m$

Once the "basic functions" d_i, s_i and ρ_i are computed, one has to compute the products of huge matrices, having a number of elements equal to $r_i r_j r_{i+j}$. For instance, for the small group $(\mathbf{Z}/2\mathbf{Z})^3$ one has $r_2 = 48$ and $r_4 = 120$. This means that the matrix representing $\Phi_{2,2}$ in this case has 276480 elements. If entries are 4-byte integers, the matrix requires around one megabyte of storage. This can grow of a factor 2 or more if entries are of a bigint type, usually involving a pointer, a length and one or more memory locations.

One can observe that when computing the cup product of a and b as $(a \otimes b) \circ \Phi_{i,j}$, as the order of the group annihilates a and b, it is enough to have an expression of $\Phi_{i,j}$ modulo the group order, or even of a divisor if one knows the exponent of the cohomology modules one is going to be working with. Thus, in the matrix computations one only needs to keep track of the results modulo a typically small integer. For groups of order at most 256, a single byte can be used for matrix entries, thus reducing the size by a factor of 4 to 12. The size of the previous matrix (for instance) becomes 270 kilobytes. The exponent of $\bigoplus_{i \neq 0} \widehat{H}^i\left((\mathbf{Z}/2\mathbf{Z})^3, \mathbf{Z}\right)$ is 2, so in this case we could even use single bit entries (provided we had an implementation of bit matrices available) and reduce the space usage to a mere 34 kilobytes.

Another thing worth noticing is that the matrix of map ρ_i is very sparse, consisting of a few one entries on the diagonal. The same is true for ρ' and ρ''. When one of those maps is in first or last position in a multiplication sequence, intermediate products have a low number of non-zero rows (respectively columns). This can be used to speed up products and reduce complexity.

As a last observation, the anticommutativity of the cup product implies that one can quickly derive $\Phi_{i,j}$, or at least a map equivalent for the cup product computation, from $\Phi_{j,i}$.

Although as general as possible, this method is computationally very heavy, especially if one doesn't have libraries implementing sparse matrices readily available.

Another method, exploited in the work of Carlson ([Car–3], [Whi]), uses composition products, which reduce the computation of cup products to the composition of maps. The main theoretical tool behind this method is the following proposition, which resumes some of the results found in Chapter VI.6 of [Br]:

Proposition III.5.2(Composition Products): *Let G be a finite group and let*

$$\varepsilon : P \longrightarrow \mathbf{Z}$$

be a complete resolution. Let M be a G-module. Then the map

$$\varepsilon \otimes \mathrm{id} : P \otimes M \longrightarrow M$$

induces an isomorphism of graded modules

$$[P, P \otimes M]_* \cong \widehat{H}^*(G, M)$$

where $[P, P \otimes M]_i$ is the set of homotopy classes of chain G-maps of degree i from P to $P \otimes M$.
Moreover, if $M = \mathbf{Z}$ with the trivial G-action, composition of chain maps corresponds to cup product in $\widehat{H}^*(G, \mathbf{Z})$.

Given a chain map f of degree $-i$

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & P_{i+2} & \longrightarrow & P_{i+1} & \longrightarrow & P_i & \longrightarrow & P_{i-1} & \longrightarrow & P_{i-2} & \longrightarrow \ldots \\
 & \downarrow f_{i+2} & & \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} & & \downarrow f_{i-2} & \\
\ldots \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & P_{-1} & \longrightarrow & P_{-2} & \longrightarrow \ldots
\end{array}$$

the corresponding cochain is f_i. In the other direction, given a G-map $a : P_i \longrightarrow \mathbf{Z}$, since P_i is projective there is a map f making the diagram

$$\begin{array}{ccccc}
 & & P_i & & \\
 & \mathrm{f} \swarrow & \downarrow a & \searrow & \\
P_0 & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0
\end{array}$$

commute. The map f can be lifted to a chain map of degree $-i$ from P to P. To compute the cup product of two elements represented by maps $a \in \mathrm{Hom}_G(P_i, \mathbf{Z})$ and $b \in \mathrm{Hom}_G(P_j, \mathbf{Z})$, one lifts a to a map from P_{i+j} to P_j and composes with b, getting a map from P_{i+j} to $\mathbf{Z}$ representing the cup product.

III.6 : Sample runs

The algorithms described in this sections were implemented as library in C++. For integer and matrix operations we relied on the LiDIA library ([Lid]), which provides functions for computing image, kernel, inverse image, Smith and Hermite Normal Forms and other functions it would be heavy to implement from scratch.

We wrote a simple program as an interface to the functionalities of the library. The source code can be found at

`http://www.beautylabs.net/software/cohomology.html`,

where we will also publish bugfixes, improvements and new results. The program lets us specify an abelian group, compute resolutions, cohomology modules and cup-products. It was run on a 600 Mhz PC with 256 Megabytes of RAM for some sets of parameters. Appendix A reports the results, including execution times, and an explanation of the output format.

Not all possible optimizations have been applied and the program still has serious limitations. Nevertheless, it allows to play with some small examples and give some insight on the beahviour of cup products. The limiting factor turns out to be the computation of the free resolutions. From Theorem III.4.1 and Chapman's results, it follows that the $\mathbf{Z}$-rank of the minimal resolution grows with the degree as a polynomial of degree the rank of the group minus 1. For cyclic groups it is constant, for groups of rank 2 is linear, for groups of rank 3 quadratic and so on. If the abelian group has rank 3 or more, then the computation gets out of reach for rather low degrees.

For cyclic groups, computations are performed very quickly, but they don't yield very interesting results.

For the other cases, we observe that the computation of the resolution for negative degrees is immediate if the corresponding positive module has been computed and the computation of the cohomology modules is very fast. All cup products we computed for negative degrees satisfy Conjecture II.8.1. In particular, for pairs of negative degrees the matrix representing the cup product has at most one non-zero element.

Appendix A: Computer experiments

A.1 : Format

This section shows the output of some computations we carried out. For some small abelian groups, the following data are shown:

- The $\mathbf{Z}$-rank of some steps $P_{-i-1}, \ldots, P_i$ of a minimal free resolution;
- The cohomology modules $\widehat{H}^j(G, \mathbf{Z})$ for $-i \leq j \leq i-1$;
- The cup products on $\widehat{H}^j(G, \mathbf{Z}) \otimes \widehat{H}^k(G, \mathbf{Z})$ for j, k and $j+k$ between $-i$ and $i-1$.

A cohomology module is represented by an array of integers, corresponding to the orders of the cyclic components. Basis of elements generating the cyclic components also stored.

If the cohomology module of degree i is represented by $(c_{i,0}, \ldots, c_{i,l(i)-1})$, then the cup product for degrees i and j is represented by a $l(i+j) \times (l(i)l(j))$ matrix $CP(i,j)$. If $0 \leq a < l(i)$ and $0 \leq b < l(j)$, then the row $al(j) + b$ of $CP(i,j)$ represents the cup product of the a-th generator of $\widehat{H}^i(G, \mathbf{Z})$, which is of order $c_{i,a}$, by the b-th generator of $\widehat{H}^j(G, \mathbf{Z})$, which has order $c_{j,b}$, in terms of the generators of $\widehat{H}^{i+j}(G, \mathbf{Z})$. The time employed in any computation is expressed in seconds.

A.2 : $\mathbf{Z}/27\mathbf{Z}$

Degree	Rank	Time
0	27	0.002
1	27	2.08
2	27	1.543
3	27	2.032
4	27	1.484
5	27	2.046
6	27	1.501
-1	27	0
-2	27	0.001
-3	27	0
-4	27	0.001
-5	27	0
-6	27	0
-7	27	0

Degree	Cohomology	Time
0	(27)	0.031
1	()	0.01
2	(27)	0.02
3	()	0.013
4	(27)	0.033
5	()	0.011
-1	()	0.021
-2	(27)	0.023
-3	()	0.015
-4	(27)	0.024
-5	()	0.014
-6	(27)	0.023

CP(0,0)
Time: 0.018
(1)

CP(-2,-2)
Time: 0.336
(1)

CP(-2,0)
Time: 0.349
(1)

CP(-2,2)
Time: 0.35
(26)

CP(2,-2)
Time: 0.267
(26)

CP(2,0)
Time: 0.271
(1)

CP(2,2)
Time: 0.399
(1)

CP(-4,-2)
Time: 0.83
(1)

CP(-4,0)
Time: 0.702
(1)

CP(-4,2)
Time: 0.674
(26)

CP(-4,4)
Time: 0.733
(1)

CP(4,-4)
Time: 0.545
(1)

CP(4,-2)
Time: 0.534
(26)

CP(4,0)
Time: 0.551
(1)

CP(-6,0)
Time: 1.055
(1)

CP(-6,2)
Time: 1.046
(26)

CP(-6,4)
Time: 1.063

(1)

A.3 : $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$

Degree	Rank	Time
0	4	0.001
1	8	0.026
2	12	0.014
3	16	0.034
4	20	0.066
5	24	0.119
6	28	0.192
7	32	0.3
-1	4	0
-2	8	0
-3	12	0
-4	16	0.001
-5	20	0
-6	24	0.001
-7	28	0.001
-8	32	0.001

Degree	Cohomology	Time
0	(4)	0.006
1	$()$	0.004
2	$(2,2)$	0.01
3	(2)	0.014
4	$(2,2,2)$	0.025
5	$(2,2)$	0.026
6	$(2,2,2,2)$	0.047
-1	$()$	0.004
-2	$(2,2)$	0.01
-3	(2)	0.012
-4	$(2,2,2)$	0.017
-5	$(2,2)$	0.02
-6	$(2,2,2,2)$	0.034
-7	$(2,2,2)$	0.039

CP(0,0)
Time: 0.012

(1)

CP(-2,-2)
Time: 0.125

$$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

CP(-2,0)
Time: 0.036

$$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$$

CP(-2,2)
Time: 0.05

$(2\ 0\ 0\ 2)$

CP(2,-2)
Time: 0.051
$\begin{pmatrix} 2 & 0 & 0 & 2 \end{pmatrix}$

CP(2,0)
Time: 0.055
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

CP(2,2)
Time: 0.128
$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

CP(-3,-3)
Time: 0.111
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

CP(-3,-2)
Time: 0.132
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

CP(-3,0)
Time: 0.032
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-3,3)
Time: 0.035
$\begin{pmatrix} 2 \end{pmatrix}$

CP(3,-3)
Time: 0.035
$\begin{pmatrix} 2 \end{pmatrix}$

CP(3,0)
Time: 0.048
$\begin{pmatrix} 1 \end{pmatrix}$

CP(3,2)
Time: 0.129
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

CP(3,3)
Time: 0.095
$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

CP(-4,-3)
Time: 0.521
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

CP(-4,-2)
Time: 0.719
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

CP(-4,0)
Time: 0.177
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

CP(-4,2)
Time: 0.167
$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

CP(-4,4)
Time: 0.436
$\begin{pmatrix} 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \end{pmatrix}$

CP(4,-4)
Time: 0.41
$\begin{pmatrix} 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \end{pmatrix}$

CP(4,-2)
Time: 0.331
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

CP(4,0)
Time: 0.27
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

CP(4,2)
Time: 0.693
$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

CP(-5,-2)
Time: 0.845
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

CP(-5,0)
Time: 0.21
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

CP(-5,2)
Time: 0.197
$\begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$

CP(-5,3)
Time: 0.112
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

CP(-5,5)
Time: 0.357
$\begin{pmatrix} 2 & 0 & 0 & 2 \end{pmatrix}$

CP(5,-5)
Time: 0.332
$\begin{pmatrix} 2 & 0 & 0 & 2 \end{pmatrix}$

CP(5,-3)
Time: 0.194
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

CP(5,-2)
Time: 0.423
$\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$

CP(5,0)
Time: 0.306
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

CP(-6,0)
Time: 0.728
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

CP(-6,2)
Time: 0.708
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

CP(-6,3)
Time: 0.291
$$\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

CP(-6,4)
Time: 0.852
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

CP(-6,6)
Time: 2.369
$$\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

CP(6,-6)
Time: 2.159
$$\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

CP(6,-4)
Time: 1.625
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CP(6,-3)
Time: 0.636
$$\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

CP(6,-2)
Time: 1.497
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CP(6,0)
Time: 0.985
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

CP(-7,0)
Time: 0.881
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CP(-7,2)
Time: 0.909
$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

CP(-7,3)
Time: 0.37
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CP(-7,4)
Time: 0.919
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

CP(-7,5)
Time: 0.703
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

A.4 : $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$

Degree	**Rank**	**Time**
0	8	0.001
1	16	0.031
2	24	0.105
3	32	0.336
4	40	0.709
5	48	1.544
6	56	2.419
7	64	4.24
-1	8	0
-2	16	0
-3	24	0.001
-4	32	0.002
-5	40	0.001
-6	48	0.002
-7	56	0.003
-8	64	0.003

Degree	**Cohomology**	**Time**
0	(8)	0.008
1	$()$	0.007
2	$(2, 4)$	0.027
3	(2)	0.026
4	$(4, 2, 2)$	0.051
5	$(2, 2)$	0.066
6	$(2, 2, 2, 4)$	0.087
-1	$()$	0.009
-2	$(2, 4)$	0.013
-3	(2)	0.018
-4	$(2, 2, 4)$	0.112
-5	$(2, 2)$	0.052
-6	$(2, 2, 2, 4)$	0.08
-7	$(2, 2, 2)$	0.104

CP(0,0)
Time: 0.014
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-2,-2)
Time: 0.5
$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&2 \end{pmatrix}$

CP(-2,0)
Time: 0.084
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(-2,2)
Time: 0.161
$\begin{pmatrix} 4&4&4&6 \end{pmatrix}$

CP(2,-2)
Time: 0.18
$\begin{pmatrix} 4&4&4&6 \end{pmatrix}$

CP(2,0)
Time: 0.166
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(2,2)
Time: 0.498
$\begin{pmatrix} 1&0&2&1 \\ 1&0&0&0 \\ 0&1&1&0 \end{pmatrix}$

CP(-3,-3)
Time: 0.564
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

CP(-3,-2)
Time: 0.662
$\begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$

CP(-3,0)
Time: 0.104
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-3,3)
Time: 0.119
$\begin{pmatrix} 4 \end{pmatrix}$

CP(3,-3)
Time: 0.127
$\begin{pmatrix} 4 \end{pmatrix}$

CP(3,0)
Time: 0.216
$\begin{pmatrix} 1 \end{pmatrix}$

CP(3,2)
Time: 0.568
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

CP(3,3)
Time: 0.328
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

CP(-4,-3)
Time: 3.226
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

CP(-4,-2)
Time: 4.494
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

CP(-4,0)
Time: 0.754
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

CP(-4,2)
Time: 0.626
$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 3 \end{pmatrix}$

CP(-4,4)
Time: 2.298
$\begin{pmatrix} 0 & 0 & 4 & 0 & 4 & 0 & 2 & 2 & 4 \end{pmatrix}$

CP(4,-4)
Time: 2.392
$\begin{pmatrix} 0 & 0 & 2 & 0 & 4 & 2 & 4 & 0 & 4 \end{pmatrix}$

CP(4,-2)
Time: 1.932
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 \end{pmatrix}$

CP(4,0)
Time: 1.324
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

CP(4,2)
Time: 3.486
$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 & 0 \end{pmatrix}$

CP(-5,-2)
Time: 5.518
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

CP(-5,0)
Time: 1.041
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

CP(-5,2)
Time: 0.825
$\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$

CP(-5,3)
Time: 0.376
$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$

CP(-5,5)
Time: 1.904
$\begin{pmatrix} 0 & 4 & 4 & 0 \end{pmatrix}$

CP(5,-5)
Time: 2.205
$\begin{pmatrix} 0 & 4 & 4 & 0 \end{pmatrix}$

CP(5,-3)
Time: 1.142
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

CP(5,-2)
Time: 2.727
$\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$

CP(5,0)
Time: 1.78
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

CP(-6,0)
Time: 4.14
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

CP(-6,2)
Time: 3.239
$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 3 \end{pmatrix}$

CP(-6,3)
Time: 1.262
$\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$

CP(-6,4)
Time: 4.236
$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 1 & 3 & 0 \end{pmatrix}$

CP(-6,6)
Time: 13.954
$\begin{pmatrix} 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 4 & 4 & 6 \end{pmatrix}$

CP(6,-6)
Time: 15.058
$\begin{pmatrix} 0 & 0 & 4 & 4 & 0 & 4 & 0 & 4 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 6 \end{pmatrix}$

CP(6,-4)
Time: 11.412
$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&2 \end{pmatrix}$$

CP(6,-3)
Time: 4.082
$$\begin{pmatrix} 0&0&0&0 \end{pmatrix}$$

CP(6,-2)
Time: 9.381
$$\begin{pmatrix} 2&0&0&0&2&2&0&2 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(6,0)
Time: 6.08
$$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

CP(-7,0)
Time: 5.133
$$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$$

CP(-7,2)
Time: 4.548
$$\begin{pmatrix} 1&0&0&0&0&1 \\ 0&0&0&1&1&0 \end{pmatrix}$$

CP(-7,3)
Time: 1.625
$$\begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&2&0 \end{pmatrix}$$

CP(-7,4)
Time: 4.521
$$\begin{pmatrix} 0&1&0&1&1&0&0&0&1 \end{pmatrix}$$

CP(-7,5)
Time: 3.573
$$\begin{pmatrix} 0&1&0&0&1&0 \\ 0&2&2&0&2&2 \end{pmatrix}$$

A.5 : $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$

Degree	**Rank**	**Time**
0	16	0.002
1	32	0.348
2	48	1.357
3	64	5.496
4	80	10.901
5	96	23.827
-1	16	0
-2	32	0.001
-3	48	0.004
-4	64	0.004
-5	80	0.007
-6	96	0.008

Degree	**Cohomology**	**Time**
0	(16)	0.016
1	()	0.015
2	(2, 8)	0.041
3	(2)	0.084
4	(2, 2, 8)	0.11
-1	()	0.009
-2	(2, 8)	0.033
-3	(2)	0.058
-4	(2, 2, 8)	0.116
-5	(2, 2)	0.188

CP(0,0)
Time: 0.008
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-2,-2)
Time: 3.583
$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&6 \end{pmatrix}$

CP(-2,0)
Time: 0.296
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(-2,2)
Time: 0.971
$\begin{pmatrix} 8&8&0&2 \end{pmatrix}$

CP(2,-2)
Time: 1.163
$\begin{pmatrix} 8&0&8&2 \end{pmatrix}$

CP(2,0)
Time: 0.871
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(2,2)
Time: 2.581
$\begin{pmatrix} 0&1&1&1 \\ 1&1&1&0 \\ 0&4&4&3 \end{pmatrix}$

CP(-3,-2)
Time: 4.549
$\begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$

CP(-3,0)
Time: 0.547
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-3,3)
Time: 1.101
$\begin{pmatrix} 8 \end{pmatrix}$

CP(3,-3)
Time: 0.935
$\begin{pmatrix} 8 \end{pmatrix}$

CP(3,0)
Time: 1.357
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-4,0)
Time: 4.671
$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$

CP(-4,2)
Time: 3.45
$\begin{pmatrix} 0&1&1&1&1&0 \\ 4&0&0&0&4&7 \end{pmatrix}$

CP(-4,4)
Time: 17.611
$\begin{pmatrix} 8&0&0&0&8&8&0&8&10 \end{pmatrix}$

CP(4,-4)
Time: 18.728
$\begin{pmatrix} 8&0&0&0&8&8&0&8&10 \end{pmatrix}$

CP(4,-2)
Time: 13.98
$\begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&6 \end{pmatrix}$

CP(4,0)
Time: 8.524
$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$

CP(-5,0)
Time: 6.354
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(-5,2)
Time: 5.356
$\begin{pmatrix} 0&1&1&1 \end{pmatrix}$

CP(-5,3)
Time: 2.644
$\begin{pmatrix} 0&1 \\ 4&0 \end{pmatrix}$

A.6 : $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$

Degree	**Rank**	**Time**
0	32	0.003
1	64	3.739
2	96	22.826
3	128	89.457
-1	32	0
-2	64	0.003
-3	96	0.012
-4	128	0.013

Degree	**Cohomology**	**Time**
0	(32)	0.042
1	$()$	0.059
2	$(4,8)$	0.191
-1	$()$	0.029
-2	$(4,8)$	0.097
-3	(4)	0.191

CP(0,0)
Time: 0.034
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-2,0)
Time: 1.591
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

CP(-2,2)
Time: 5.231
$\begin{pmatrix} 24 & 0 & 0 & 4 \end{pmatrix}$

CP(2,-2)
Time: 8.9
$\begin{pmatrix} 24 & 0 & 0 & 4 \end{pmatrix}$

CP(2,0)
Time: 6.134
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

CP(-3,0)
Time: 3.241
$\begin{pmatrix} 1 \end{pmatrix}$

A.7 : $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$

Degree	Rank	Time
0	9	0.001
1	18	0.086
2	27	0.409
3	36	1.328
4	45	2.712
5	54	5.962
6	63	9.605
0	9	0
−1	9	0
−2	18	0.001
−3	27	0.001
−4	36	0.001
−5	45	0.002
−6	54	0.002
−7	63	0.003

Degree	Cohomology	Time
0	(9)	0.039
1	()	0.025
2	(3, 3)	0.092
3	(3)	0.115
4	(3, 3, 3)	0.252
5	(3, 3)	0.385
−1	()	0.019
−2	(3, 3)	0.045
−3	(3)	0.076
−4	(3, 3, 3)	0.114
−5	(3, 3)	0.181
−6	(3, 3, 3, 3)	0.265

CP(0,0)
Time: 0.031
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-2,-2)
Time: 2.302
$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$

CP(-2,0)
Time: 0.253
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(-2,2)
Time: 0.685
$\begin{pmatrix} 0&6&6&0 \end{pmatrix}$

CP(2,-2)
Time: 0.828
$\begin{pmatrix} 0&6&6&0 \end{pmatrix}$

CP(2,0)
Time: 0.625
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(2,2)
Time: 2.118
$\begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 2&1&1&2 \end{pmatrix}$

CP(-3,-3)
Time: 2.596
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

CP(-3,-2)
Time: 2.985
$\begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$

CP(-3,0)
Time: 0.391
$\begin{pmatrix} 1 \end{pmatrix}$

CP(-3,3)
Time: 0.456
$\begin{pmatrix} 3 \end{pmatrix}$

CP(3,-3)
Time: 0.533
$\begin{pmatrix} 6 \end{pmatrix}$

CP(3,0)
Time: 1.027
$\begin{pmatrix} 1 \end{pmatrix}$

CP(3,2)
Time: 2.709
$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

CP(-4,-2)
Time: 20.209
$\begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$

CP(-4,0)
Time: 3.209
$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$

CP(-4,2)
Time: 2.458
$\begin{pmatrix} 0&2&0&0&2&1 \\ 0&0&2&0&1&2 \end{pmatrix}$

CP(-4,4)
Time: 11
$\begin{pmatrix} 0&3&0&3&0&0&0&0&3 \end{pmatrix}$

CP(4,-4)
Time: 12.096
$\begin{pmatrix} 0&3&0&3&0&0&0&0&3 \end{pmatrix}$

CP(4,-2)
Time: 8.605
$\begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$

CP(4,0)
Time: 6.951
$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$

CP(-5,0)
Time: 4.33
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(-5,2)
Time: 3.028
$\begin{pmatrix} 0&2&2&0 \end{pmatrix}$

CP(-5,3)
Time: 1.591
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(-5,5)
Time: 9.711
$\begin{pmatrix} 6&0&6&3 \end{pmatrix}$

CP(5,-5)
Time: 10.063
$\begin{pmatrix} 3&3&0&6 \end{pmatrix}$

CP(5,-3)
Time: 5.446
$\begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$

CP(5,-2)
Time: 11.398
$\begin{pmatrix} 0&0&0&0 \end{pmatrix}$

CP(5,0)
Time: 8.41
$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$

CP(-6,0)
Time: 20.079
$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$

CP(-6,2)
Time: 12.892
$\begin{pmatrix} 0&0&0&0&0&1&1&0 \\ 1&0&0&0&0&0&0&1 \\ 0&0&1&2&0&0&2&2 \end{pmatrix}$

CP(-6,3)
Time: 5.169
$\begin{pmatrix} 0&0&0&0 \end{pmatrix}$

CP(-6,4)
Time: 18.062
$\begin{pmatrix} 0&0&0&0&0&1&0&2&0&2&0&1 \\ 2&0&0&0&0&2&0&0&0&0&2&1 \end{pmatrix}$

A.8 : $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$

Degree	**Rank**	**Time**
0	8	0.001
1	24	0.109
2	48	0.408
3	80	3.25
4	120	15.152
5	168	60.205
6	224	188.211
-1	8	0
-2	24	0.001
-3	48	0.003
-4	80	0.015
-5	120	0.021
-6	168	0.042
-7	224	0.081

Degree	**Cohomology**	**Time**
0	(8)	0.011
1	$()$	0.019
2	$(2,2,2)$	0.097
3	$(2,2,2)$	0.309
4	$(2,2,2,2,2,2,2)$	0.884
5	$(2,2,2,2,2,2,2,2)$	2.272
-1	$()$	0.009
-2	$(2,2,2)$	0.043
-3	$(2,2,2)$	0.104
-4	$(2,2,2,2,2,2,2)$	0.312
-5	$(2,2,2,2,2,2,2,2)$	0.935
-6	$(2,2,2,2,2,2,2,2,2,2,2,2,2)$	2.024

CP(0,0)
Time: 0.021

$$\begin{pmatrix} 1 \end{pmatrix}$$

CP(-2,-2)
Time: 9.463

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(-2,0)
Time: 0.223

$$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$$

CP(-2,2)
Time: 0.926

$$\begin{pmatrix} 4&0&0&0&4&0&0&0&4 \end{pmatrix}$$

CP(2,-2)

Time: 1.578

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{pmatrix}$$

CP(2,0)

Time: 1.12

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CP(2,2)

Time: 7.309

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

CP(-3,-3)

Time: 170.802

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CP(-3,-2)

Time: 49.05

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CP(-3,0)

Time: 1.305

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CP(-3,3)

Time: 5.751

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 4 \end{pmatrix}$$

CP(3,-3)

Time: 8.952

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 4 \end{pmatrix}$$

CP(3,0)

Time: 5.456

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CP(3,2)

Time: 31.419

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

CP(-4,-2)

Time: 490.214

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CP(-4,0)
Time: 13.807

$$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$$

CP(-4,2)
Time: 7.041

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\ 0&1&1&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0 \end{pmatrix}$$

CP(-4,4)
Time: 141.238

(0 0 0 0 0 0 4 0 0 0 0 4 0 0 0 0 0 0 0 4 0 0 4 0 0 0 4 0 0 0 4 0 0 0 0 0 0 0 4 0 0 0 4 0 0 0 0 0 0)

CP(4,-4)
Time: 208.297

(0 0 0 0 0 0 4 0 0 0 4 0 0 0 0 0 0 0 4 0 0 0 0 0 0 0 4 0 0 4 0 0 0 0 0 0 0 4 4 0 0 0 4 0 0 0 0 0 0)

CP(4,-2)
Time: 105.299

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(4,0)
Time: 55.026

$$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$$

CP(-5,0)
Time: 65.574

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

CP(-5,2)
Time: 27.017

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&1&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&1&1&0&0&0&0&0&1 \end{pmatrix}$$

CP(-5,3)
Time: 23.824

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&1&1&0&0&0\\ 0&0&0&0&1&1&0&1&0&0&0&0&1&0&1&1&0&1&0&0&0&1&0&1 \end{pmatrix}$$

CP(5,-5)
Time: 1013.75

(0 0 0 0 0 0 0 4 0 0 0 0 4 0 0 4 0 4 0 0 0 0 0 0 0 0 0 0 0 4 0 0 0 0 0 4 0 0 0 0 0 0 4 0 0 0 0 0 4 0 0 0 0 0 0 0 0 0 0 0 4 4 4 0)

CP(5,-3)
Time: 450.157

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(5,-2)
Time: 519.151

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(5,0)

Time: 266.014

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

CP(-6,0)

Time: 408.078

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

CP(-6,3)

Time: 84.721

$$\begin{pmatrix} 0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&1&1&0&0\\ 0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(-6,4)

Time: 322.637

$$\begin{pmatrix} 0\,000000000000100000000100000000000000010100000000000010000100000101000010100000000000000000\\ 0\,000000000101000001000000000000000101010100000000000110000000000101010010101000101000000000\\ 1\,000000000101110000001100000100000111000000010000100000000100101100000110000000101011000000 \end{pmatrix}$$

A.9 : $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$

Degree	**Rank**	**Time**
0	27	0.009
1	81	2.464
2	162	54.192
3	270	539.612
4	405	2607.23
5	567	10918.7
-1	27	0
-2	81	0.071
-3	162	0.051
-4	270	0.108
-5	405	0.348
-6	567	4.708

Degree	**Cohomology**	**Time**
0	(27)	1.864
1	$()$	0.374
2	$(3,3,3)$	3.066
3	$(3,3,3)$	9.045
4	$(3,3,3,3,3,3,3)$	29.763
-1	$()$	0.035
-2	$(3,3,3)$	0.249
-3	$(3,3,3)$	1.259
-4	$(3,3,3,3,3,3,3)$	2.696
-5	$(3,3,3,3,3,3,3,3)$	25.118

CP(0,0)
Time: 0.336

$\begin{pmatrix} 1 \end{pmatrix}$

CP(-2,-2)
Time: 1626.62

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(-2,0)
Time: 5.578

$$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$$

CP(-2,2)
Time: 66.249

$\begin{pmatrix} 0 & 18 & 0 & 18 & 0 & 0 & 0 & 0 & 18 \end{pmatrix}$

CP(2,-2)
Time: 186.352

$\begin{pmatrix} 0 & 18 & 0 & 18 & 0 & 0 & 0 & 0 & 18 \end{pmatrix}$

CP(2,0)
Time: 73.576

$$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$$

CP(2,2)
Time: 419.125

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 1&1&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 2&0&1&0&0&0&1&0&2 \\ 0&0&0&0&1&2&0&2&1 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(-3,-2)
Time: 9823.51

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(-3,0)
Time: 82.091

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CP(-3,3)
Time: 507.553

$$\begin{pmatrix} 0 & 9 & 0 & 9 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}$$

CP(3,-3)
Time: 1830.78

$$\begin{pmatrix} 0 & 18 & 0 & 18 & 0 & 0 & 0 & 0 & 18 \end{pmatrix}$$

CP(3,0)
Time: 809.65

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CP(-4,0)
Time: 1754.43

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CP(-4,2)
Time: 279.043

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0 \end{pmatrix}$$

A.10 : $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$

Degree	**Rank**	**Time**
0	16	0.002
1	64	0.373
2	160	16.684
3	320	318.975
4	560	3429.79
-1	16	0
-2	64	0.008
-3	160	0.063
-4	320	0.192
-5	560	1.22

Degree	**Cohomology**	**Time**
0	(16)	0.093
1	$()$	0.158
2	$(2, 2, 2, 2)$	1.297
3	$(2, 2, 2, 2, 2, 2)$	11.061
-1	$()$	0.018
-2	$(2, 2, 2, 2)$	0.181
-3	$(2, 2, 2, 2, 2, 2)$	1.189
-4	$(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$	8.693

CP(0,0)
Time: 0.085

$$\begin{pmatrix} 1 \end{pmatrix}$$

CP(-2,-2)
Time: 1557.43

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

CP(-2,0)
Time: 2.211

$$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

CP(-2,2)
Time: 29.758

$$\begin{pmatrix} 0&0&8&0&8&0&0&0&0&8&0&0&0&0&0&8 \end{pmatrix}$$

CP(2,-2)
Time: 81.879

$$\begin{pmatrix} 0&8&0&0&0&0&8&0&8&0&0&0&0&0&0&8 \end{pmatrix}$$

CP(2,0)
Time: 30.595

$$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

CP(-3,0)
Time: 49.51

$$\begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$$

CP(-3,3)
Time: 1550.69

$$\begin{pmatrix} 0&0&0&0&0&8&8&0&0&0&0&0&0&8&0&0&0&0&0&0&0&8&0&0&0&0&8&0&0&0&0&0&8&0&8&0 \end{pmatrix}$$

CP(3,-3)
Time: 3593.18

$$\begin{pmatrix} 0&8&0&0&0&0&0&0&8&0&0&0&0&0&0&0&8&8&0&0&0&8&0&0&0&0&0&0&0&8&8&0&0&0&0&0 \end{pmatrix}$$

CP(3,0)
Time: 810.336

$$\begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$$

CP(-4,0)

Time: 1853.6

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

CP(-4,2)

Time: 260.69

$$\begin{pmatrix}
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&1&1&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0
\end{pmatrix}$$

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